Matching Through Institutions

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Abstract

In a housing allocation problem where apartments are assigned to some household through an institution, we introduce a new assignment procedure, the Nested Deferred Acceptance (NDA) algorithm. If institutions do not face distributional constraints, the NDA is fair, Pareto undominated by any fair mechanism and strategy-proof. We analyse the consequences of imposing distributional constraints. We first show that an assignment respecting the constraints does not always exist. Second, we show that the NDA algorithm does not necessarily output fair matchings. We introduce an over-demand condition to guarantee the existence of assignments respecting distributional constraints, and we identify interrupters who are the origin of the lack of fairness, as in Kesten (2010), We introduce a new algorithm – NDA with interrupters – which preserves the properties of the NDA algorithm under the over-demand condition.

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1 Introduction

This paper studies matching markets intermediated by institutions. Institutions own objects or have priority rights over objects. Agents are attached to institutions and have preferences over the entire set of objects. In this setting, at minimum, agents can be matched to objects belonging to their institutions. We consider a more flexible assignment rule where institutions pool their resources in order to increase the choice sets of all agents. We investigate two possibilities. In the first one (absence of distributional constraints), institutions do not place any restriction on the number of assignments they control. The final matching is unrestricted and the assignment problem becomes similar to a classical school choice problem. In the second one (matching with distributional constraints), each institution places a quota on the number of assignments they control in the final matching, and the problem belongs to the family of assignment problems with distributional constraints like controlled school choice, programs of student or tuition exchange, or the assignment of doctors to regional hospitals in Japan.

A prime example of matching through institutions – and the one which motivated our study – is the assignment of social housing in Paris.[1] Four different institutions have the right to assign social housing units: the Ministry of Housing, the local mayors of Paris districts, the central mayor of Paris, and a representative of private institutions involved in the construction of social housing. Every month, a committee meets to assign vacant apartments to candidates which are chosen by the four institutions. Each institution has priority rights over apartment buildings which were built with their contribution. However, candidates from different institutions may end up occupying apartments belonging to other institutions. The final assignment must respect quotas by institutions 30 % to candidates from the Ministry of Housing, 30% to candidates from the mayors of districts, 20 % to candidates on the list of the central city hall and the remaining 20 % to candidates from private institutions.

Another example of matching through institutions is the assignment of study abroad programs in American universities. Faced with escalating costs of maintaining facilities abroad, universities increasingly offer access to their programs to students from other institutions.(In some cases, like the University of California Education Abroad Program or the Universities Study Abroad

[1]See the annual report (in French) on assignment of social housing in Paris in APUR (2014).

Consortium, resources are pooled in a centralized way.) Universities typically grant priority access to their own students. By opening access, they increase the choice set of students. Formal quotas for students are rarely enforced, but the universities aim at an overall balance between outgoing and incoming students in study abroad programs. In this paper, we model matching through institutions as a three-sided market involving agents (which we refer to as "households"), objects (which we refer to as "apartments") and institutions. Households have preferences over apartments. Institutions set priorities over pairs assigning apartments to households. For each apartment, a priority ranking indicates the assignment rights of each institution. We consider two versions of the market: one without distributional constraints, and one where each institution has a fixed quota on the number of apartments which are assigned to the households attached to it.

The main contribution of the paper is to propose a new algorithm for matching through institutions, the Nested Deferrence Acceptance (NDA) algorithm. The NDA combines two different deferred acceptance algorithms. During the first one (the "outer loop" in the algorithm), each household asks for her most preferred apartment among those which have not yet rejected her. Given this list of demands, we run a second deferred acceptance algorithm (the "inner loop") among institutions. Each institution chooses a set of apartments that maximizes its preference and does not exceed its vector of quotas. If more than one institution is interested in one apartment, ties are broken according to the priority assignment rights of each apartment. The process is repeated until apartments are assigned to households in such a way that institutions respect their quotas (In the model without distributional constraints, this only requires one step in the algorithm). Going back to the outer loop, we next ask rejected households to apply for their next preferred apartment and the procedure continues until no household is rejected.

We first analyze the properties of the NDA in a model without distributional constraints. We show that, in our extended model, the NDA inherits the properties of the deferred acceptance algorithm in classical school choice problems: it results in fair allocations, in a matching which is not Pareto dominated by any other fair matching, and is strategy-proof.

We then consider the more interesting case where institutions have fixed quotas. We first show that assignments satisfying distributional constraints may fail to exist and we provide sufficient condition – the over-demand condition under which assignments exist and the NDA produces an assignment satisfying the constraints. However, even under the over-demand condition, we observe that the NDA does not necessarily output a fair outcome. As in Kesten (2010), the problem comes from the presence of "interrupters" institutions which are temporarily assigned to apartments that they will not be assigned to in the final matching, thereby blocking access of apartments to households from other institutions.

The NDA with Interrupters (NDAI) improves the NDA by dropping apartments from the preferences of interrupter institutions. Our main result shows that the output given by the NDAI is fair over households of the same institution, Pareto undominated by any other fair assignment satisfying the distributional constraint and strategy-proof. (Recall that, by contrast, the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) of Kesten is manipulable because interrupters are students who strategically report their preferences while in our case interrupters are institutions which do not submit preferences.)

Even though the two models are not identical, our matching market bears strong similarities to the matching with distributional constraints studied in Kamada and Kojima (2014a), (2014b) and (2015). Inspired by the assignment of doctors to hospitals with flexible regional quotas, Kamada and Kojima (2014a), (2014b) and (2015) propose a very general model where matching can be flexible across hospitals in the same region, and regions have preferences over the matching of doctors to hospitals in their jurisdictions. We show that the NDAI algorithm can be adapted to matching with distributional constraints, by reinterpreting regions as "institutions", doctors as "households" and jobs in hospitals as "apartments".

For the most part of the analysis, we suppose that each household is attached to a single institution: each institution has a separate list of candidates. In the assignment of social housing in Paris, this condition is not necessarily satisfied and the same household can be proposed by different institutions. In an appendix, we analyze the more general case where agents can be attached to more than one institution. We first note that matchings satisfying distributional constraints may then fail to exist, as it may be impossible to assign households to institutions in such a way to satisfy the quotas. Even when matchings satisfying distributional constraints exist, the NDAI may not produce fair assignments.

The problem we consider is related to the rapidly emerging literature on matchings with constraints. In the school choice context, a number of recent papers have analyzed the effect of constraints resulting from affirmative action considerations. One stream of papers interprets affirmative action as "leveling the playing field", as in Kojima (2012) and Hafalir, Yenmez and Yildirim (2013) and Kominers and Sönmez (2013). Another stream of papers closer to our motivation consider affirmative action is an objective per se, formalized either by the existence of quotas as in Abdulkadiroğlu (2003), or bounds as in Ehlers, Hafalir, Yenmez and Yildirim (2014) and Fragiadakis and Troyan (2015). One of the objective of these papers is to refine stability concepts and the deferred acceptance algorithms to conform to the bounds and quotas. Echenique and Yenmez (2015) and Erdil and Kumano (2012) consider diversity as an objective of the school district and explore the tension between diversity objectives, stability and efficiency of priority systems and matching rules.

Our analysis also bears a close connection to exchange markets with balanced constraints recently studied by Dur and Ünver (2014) and Biró, Klijn and Pápai (2015). These papers model exchange programs (like the "tuition exchange" for children of faculty members or the "Erasmus exchange program" in European universities) where a balance must be kept between the number of incoming and outgoing students. Both papers consider a two-sided (rather than three-sided) matching problem, where colleges have preferences over students, students preferences over colleges, In Dur and Ünver (2014)'s tuition exchange model, students are ranked inside each college according to an exogenous priority (for example, tenure of faculty member). Dur and Ünver (2014) emphasize the balancedness condition – which, strictly applied, imposes that the number of incoming and outgoing students in each college must be equal. As in our analysis with distributional constraints, they show that the balancedness condition may lead to impossibility results, when associated with different natural axioms, like individual rationality or fairness. Their analysis focuses on efficiency and they propose a new procedure based on the Top Trading Cycle algorithm (rather than Deferred Acceptance). Biró, Klijn and Pápai (2015) also focus attention on an extension of the TTC algorithm to analyze student exchange programs where a balancedness condition holds.

Finally, we note that the assignment of social housing that motivated our study has recently been analyzed in a series of papers (Leshno (2014), Bloch and Cantala (2015), Schummer (2015) and Thakral (2015)) which focus on very different aspects of the problem – the revelation of persistent information on types in Leshno (2014), the dynamic sequence of decisions in Bloch and Cantala (2014), the manipulation of orders in Schummer (2015) and multiple waiting list mechanisms in Thakral (2015).

The present work is organized as follows. Section 2 introduces formally the model. Section 3 presents the Nested DA and the main results about it when there are no distributional constraints. Section 4 discusses the consequences of introducing distributional constraints. In Section 5 we show the relation between the matching market through institution with a market with regional preferences. The proofs of all lemmas and propositions in the text are collected in Appendix A and Appendix B contains the results for the case where agents can be attached to multiple institutions.

2 The Model

A matching market with institutions is an 8−tuple $(I, Q, H, \tau, A, P, \succ, \pi)$ where:

1. $I = \{1, 2, ..., N\}$ is the finite set of institutions, a generic institution is i;

2. $Q = (q_i)_{i=1}^N$ is the vector of quotas, where q_i is the quota of institution i, a generic quota is q;

3. $H = \{h_1, ..., h_H\}$ is the finite set of households, a generic household is h;

4. $\tau : H \to I$ is the type function, which assigns to every household an institution $\tau(h)$. Conversely, $H_i = \{h \in H \mid i = \tau(h)\}$ is the set of agents attached to institution i.[2]

5. $A = \{a_1, ..., a_A\}$ is the finite set of apartments, a generic apartment is a;

6. $P = (P_{h_1}, ..., P_{h_H})$ is the vector of households' preferences, P_h is the strict preferences of household $h \in H$ over $A \cup \{\varnothing\}$; aP_ha' means that households h prefers a to a', an apartment a is acceptable for household h if $aP_h\varnothing$. Let $P_h : a_{1_h}, a_{2_h}, \dots a_{A_h}$, and R_h be the antisymmetric preference list where aR_hb and bR_ha if and only if $a = b$;

[2] We assume that every agent is attached to a single institution. In an extension discussed in Subsection 4.3, we also consider the case where the same individual can be attached to two or more institutions – a situation which sometimes arises in the context of social housing where the same household can belong to the applicants' pool of different institutions.

7. $\pi = (\pi_a)_{a \in A}$ is the profile of institutions' priorities over apartments; π_a is the priority of apartment a over institutions $i \in I$. Let $A(i) = \{a | i\pi_a i' \text{ for all } i' \in I\}$ be the set of apartments where institution i has top priority. If institutions own apartments, $A(i)$ denotes the set of apartments owned by institution i.

8. For all $i \in I$, $\succ^i$ is the preference of institution i on the elements in $2^{A \times H_i}$. Each institution has preferences over the assignment of apartments to households. This general formulation allows us to consider different interpretations. In a first interpretation, institutions only care about households attached to them, and have preferences inherited from the preferences of households. In a second interpretation, institutions only care about the apartments they own, and have preferences over the households occupying the apartments they own. In a third interpretations, institutions care about the complementarity between households and apartments, for example giving higher priority to large households for large apartments. We write $(a, h) \succ^i \varnothing$ if the pair (a, h) is acceptable for institution i.

   We assume that for all $i \in I$ the preference $\succ^i$ is responsive on elements in $2^{A \times H_i}$, i.e. for $U \in 2^{A \times H_i}$ and pairs (a_r, h_r), $(a_s, h_s) \in (A \times H_i) \setminus U$ we have that

   i. $U \cup (a_r, h_r) \succsim^i U \cup (a_s, h_s)$ if and only if $(a_r, h_r) \succsim^i (a_s, h_s)$, and

   ii. $U \cup (a_r, h_r) \succ^i U$ if and only if $(a_r, h_r) \succ^i \varnothing$; A particular case is a lexicographic preference $\succ^i = (\succ^i_A, (\succ^i_a)_{a \in A})$ where $\succ_A$ is the preference of i over apartments, and $\succ^i_a$ is the preference of i over households for each $a \in A$. So, for all pairs $(a_r, h_r), (a_s, h_s) \in A \times H$ we have that

$$(a_r, h_r) \succ^i (a_s, h_s) \text{ if and only if } \begin{cases} a_r \succ^i_A a_s & \text{if } a_r \neq a_s, \\ h_r \succ^i_{a_r} h_s & \text{otherwise.} \end{cases}.$$

When matching markets are intermediated by institutions, matchings are not simply defined as mappings between households and apartments. The mechanism we propose consists in two separate assignments: we first tentatively assign apartments to institutions, then assign households to pairs consisting of one apartment and its matched institution. We thus formalize an assignment as: (i) a many-to-one matching between apartments and institutions, and (ii) a one-to-one matching between households and pairs composed by one apartment and one institution. Such assignments are formalized in the following definition.

An **assignment** $\mu = (\theta, \varphi)$ is a pair such that:

i. $\theta : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ where

   i.a $\theta(a) \in I \cup \{\varnothing\}$,

   i.b $\theta(i) \in 2^A$ and $|\theta(i)| \leq q_i$,

   i.c $a \in \theta(i)$ if and only if $\theta(a) = i$;

ii. $\varphi : (A \times I) \cup H \to (A \times I) \cup H \cup \{\varnothing\}$, where

   ii.a $\varphi(h) \in A \times I \cup \{\varnothing\}$,

   ii.b $\varphi(a, i) \in H \cup \{\varnothing\}$,

   ii.c $\varphi(h) = (a, i) \Leftrightarrow \varphi(a, i) = h$. The corresponding projections are $\varphi_A(h) = a$ and $\varphi_I(h) = i$;

iii. $\theta(a) = i$ if and only if $\varphi(h) = (a, i)$ for some $h \in H$.

Conditions i. a, b and c define the many-to-one matching θ between apartments and institutions, taking into account the distributional constraint imposed by the quotas per institution. Conditions ii. a, b and c define the one-to-one matching φ between households and pairs composed by one apartment and one institution. Condition 3 defines a consistency condition between the two matchings, by requiring that whenever a household is assigned to a pair consisting of an apartment and an institution in φ, the apartment and institution are assigned to each other in θ.

The match of $h \in H$ is $\varphi(h) \in (A \times I) \cup \{\varnothing\}$, h is unmatched if $\varphi(h) = \varnothing$. The assignment of i is $\mu(i) = \{(a, h) \in A \times H : a \in \theta(i) \text{ and } \varphi(h) = a\}$.

We illustrate the definition of the assignments θ and φ with the following example. Consider $(I, Q, H, \tau, A, P, \succ, \pi)$ a market with institutions where $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$,

$A = \{a_1, a_2, a_3, a_4\}$, a vector of priorities $\succ^i$, a vector of preferences P and a profile of priorities π. The vector of quotas is $Q = (1, 2, 1)$, and the type function is given by

$$H_1 = \{h_3\}, H_2 = \{h_4, h_5\}, H_3 = \{h_1, h_2\}.$$

A typical assignment for this market is represented as follows

$$\mu = \underbrace{\left(\begin{array}{ccccccc} h_3 & h_5 & h_1 & \varnothing & h_2 & h_4 & \varnothing \\ \overbrace{\hphantom{a_2}}^{\varphi(h_3)} & \overbrace{\hphantom{a_4}}^{\varphi(h_5)} & \overbrace{\hphantom{a_3}}^{\varphi(h_1)} & \varnothing & \overbrace{\hphantom{\varnothing}}^{\varphi(h_2)} & \overbrace{\hphantom{\varnothing}}^{\varphi(h_4)} & \overbrace{\hphantom{a_1}}^{\varphi(\varnothing)} \\ a_2 & a_4 & a_3 & \underbrace{\hphantom{\varnothing}}_{\varphi^{-1}(\varnothing)} & \varnothing & \varnothing & a_1 \\ 2 & 2 & 3 & 1 & \varnothing & \varnothing & \varnothing \end{array} \right)}_{\theta(2)\ \ \theta(3)\ \ \theta(1)\ \ \theta^{-1}(\varnothing)}.$$

We now define classical properties of the assignment μ for matching with institutions. An assignment μ is **individually rational** if

i. for all $h \in H$ either $\varphi_A(h) P_h \varnothing$ or $\varphi(h) = \varnothing$, and

ii. for all $i \in I$, either $(a, h) \succ^i \varnothing$ for all $(a, h) \in \mu(i)$, or $\mu(i) = \varnothing$.

We assume that the set of individually rational assignments is nonempty.

Consider $\mathcal{U} = \{U \in 2^{A \times H} |$neither households nor apartments are paired twice in $U\}$. The set $\mathcal{U}$ collects pairs of apartments and households such that every apartment and household only appear in one of the pairs. For any institution $i \in I$, we define the **choice function** Ch_i as a mapping choosing the pairs with the highest preference for i in $\mathcal{U}$: for all $(U, q_i) \in 2^{A \times H} \times \mathbb{Z}_+$, the choice of i is the set $Ch_i(U, q_i) = \max_{\succ^i}\{u \subseteq U \mid (|u| \leq q_i) \text{ and } u \in \mathcal{U}\}$.

An assignment μ is **non-wasteful** if no household-institution pair (h, i) can claim an empty apartment a, i.e. there is no i, h and a such that:

i. $a P_h \varphi_A(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i)$, and

iii. $\theta(a) = \varnothing$.

A household-institution pair (h, i) has **justified envy** over the household-institution pair (h', i') at the individually rational assignment μ if $i = \tau(h)$, $i' = \tau(h')$ and there exists $\varphi_A(h') = a \in \theta(i')$, such that

i. $aP_h\varphi_A(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup (a, h), q_i)$, and

iii. $i\pi_a i'$.

There is **justified envy over households of the same type** when a pair (h, i) has justified envy over a pair (h', i).

An assignment μ is **fair** if it is individually rational, non-wasteful and there is no justified envy. A matching μ is **fair over households of the same type** if it is individually rational, non-wasteful and there is no justified envy for households of the same type.

An assignment μ is **Pareto efficient** if there is no matching μ' such that all households prefer μ' to μ, with strict inequality for at least one household. An assignment μ' **Pareto dominates** another assignment μ if $\mu'(h)R_h\mu(h)$ for each $h \in H$, and $\mu'(h')P_{h'}\mu(h')$ for at least one $h' \in H$.

A **mechanism** Λ associates a profile of preference lists with an assignment μ. Let R_h be the true preference list of each household h. The set of all possible preference lists of household h is denoted by $\Re_h$. A profile of preference list is a vector $R' = (R'_{h_1}, R'_{h_2}, \ldots, R'_{h_H}) \in \Re_{h_1} \times \Re_{h_2} \times \cdots \times \Re_{h_H} = \Re$. As usual, R_{-h} is the profile of all preference list except R_h.

A mechanism Λ is **strategy proof for households** if telling the truth is a dominant strategy for all households, i.e.

$$\Lambda[R_h, R_{-h}](h)R_h\Lambda[R'_h, R_{-h}](h) \text{ for all } R'_h \in \Re_h \text{ and } R_{-h} \in \Re_{-h}.$$

3 The Nested Deferred Acceptance Mechanism

We introduce the Nested Deferred Acceptance (NDA), which ouputs an assignment $\mu = (\theta, \varphi)$ in the matching market with institutions. The idea behind this assignment procedure is to compute simultaneously a many-to-one matching, θ, and a one-to-one matching, φ, by nesting two deferred acceptance algorithms. In the main DA iteration (the "outer loop"), each unassigned household asks for her most preferred apartment. Given these demands, we run another DA (the "inner loop") where each institution demands a pair of apartments and households. The procedure is then iterated. Formally, the NDA proceed as follows:

Initialization

Consider a market $(I, Q, H, \tau, A, P, \succ, \pi)$. The assignment is initialized to be the empty assignment, so $\mu^0(i) = \mu^0(a) = \mu^0(h) = \varnothing$, i.e. $\theta^0(i) = \theta^0(a) = \varphi^0(h) = \varnothing$ for all $i \in I$, $a \in A$, $h \in H$.

Let $A_h^t = A$ and $t := 1$.

A. Eliciting the demand of households (the outer loop)

All unassigned households h ask for their most preferred apartment in A_h^t, denoted by D_h^t, while matched households h' iterate their demand to their match, i.e. $D_{h'}^t = \{\varphi_A^{t-1}(h')\}$.

For all $i \in I$ and $a \in A$, we define the set of households of type i that demand apartment a as follows:

$$H_{a,i}^t = \{h \in H \mid D_h^t = \{a\} \text{ and } i = \tau(h)\}.$$

The set of pairs (a, h) that can be assigned to institution i is defined as

$$M_i^t = \{(a, h) \in A \times H \mid (a, h) \succ^i \varnothing \text{ and } h \in H_{a,i}^t\}.$$

B. Iteration over $\mathbf{M_i^t}$ to match the demands of institutions and apartments (the inner loop)

Let $\theta^0(i) = \varnothing$ for all $i \in I$.

Let $\tilde{M}_i^s = M_i^t$ and $s := 1$.

B.1 All institutions i demand the set of pairs $Ch_i(\tilde{M}_i^s, q_i)$. So, the set of institutions that demand an apartment a is

$$I_a^s = \{i \in I \mid \text{ there exists } (a, h) \in Ch_i(\tilde{M}_i^s, q_i)\}.$$

B.2 For all apartments a such that $I_a^s \neq \varnothing$, apartment a is assigned to the institution with the highest priority under π_a, i.e. $a \in \theta^s(i)$ if and only if $i = \max_{\pi_a} I_a^s$.

For all institutions i, let $\tilde{M}_i^{s+1} := \tilde{M}_i^s \setminus \{(a, h) \in \tilde{M}_i^s \mid (a, h) \in Ch_i(\tilde{M}_i^s, q_i) \text{ and } a \notin \theta^s(i)\}$. That is to say, we delete from the set $\tilde{M}_i^s$ those pairs where the institution i is rejected.

If $|\theta^s(i)| = q_i$ for all institutions i, or $\tilde{M}_i^{s+1} = \varnothing$ for all institutions i for which $|\theta^s(i)| < q_i$, go to B.3; otherwise, let $s = s + 1$ and go to B.1.

B.3 Rename $\theta^t(i) := \theta^S(i)$ where S is the last iteration of B.1. Furthermore, each pair (a, i) is tentatively assigned to household h if and only if $a \in \theta^t(i)$ and $(a, h) \in Ch_i(\tilde{M}_i^S, q_i)$. That is to say $\varphi^t(h) = (a, i)$.

C. Iteration over $\mathbf{D_h^t}$

For all unassigned household, h, let $A_h^{t+1} := A_h^t \diagdown \{\max_{P_h} A_h^t\}$, $t := t + 1$. If each household has been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the output assignment. Otherwise, go to A.

The output of the previous mechanism depends on the market $E = (H, A, P, I, \succ, \pi_A, Q)$. So, it is denoted by $\mu^{NDA}[E] = (\theta^{NDA}[E], \varphi^{NDA}[E])$, or simply $\mu^{NDA} = (\theta^{NDA}, \varphi^{NDA})$ whenever there is no confusion. We use $NDA[P]$ to denote the NDA algorithm under the preference profile P. Note that the NDA algorithm has a finite number of steps because each DA ends in polynomial time.

3.1 An example of NDA

The following example shows how the NDA algorithm works.

Example 3.1. Consider the market $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $A = \{a_1, a_2, a_3\}$. The vector of quotas is $Q = (3, 3, 3)$, and the type function is defined as

$$\tau^{-1}(1) = \{h_1, h_2, h_6\},\ \tau^{-1}(2) = \{h_3, h_4\},\ \tau^{-1}(3) = \{h_5\}.$$

The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 & \succ^3 \\ (a_1, h_1) & (a_1, h_3) & (a_1, h_5) \\ (a_1, h_2) & (a_1, h_4) & (a_2, h_5) \\ (a_2, h_1) & (a_2, h_3) & (a_3, h_5) \\ (a_2, h_2) & (a_2, h_4) & \\ (a_3, h_6) & (a_3, h_3) & \\ (a_3, h_2) & (a_3, h_4) & \\ & & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} & P_{h_6} \\ a_1 & a_2 & a_1 & a_2 & a_1 & a_3 \\ a_2 & a_1 & a_2 & a_1 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_1 \end{pmatrix}.$$

The profile of apartments priorities is

$$\pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Running the NDA algorithm, the stage A at step 1 is summarized in Table 1.

I	$H^1_{a_1,i}$	$H^1_{a_2,i}$	$H^1_{a_3,i}$	$M^1_i = \tilde{M}^0_i$	Ch^1_i
1	h_1	h_2	h_6	$(a_1, h_1), (a_2, h_2), (a_3, h_6)$	$(a_1, h_1), (a_2, h_2), (a_3, h_6)$
2	h_3	h_4	$\varnothing$	$(a_1, h_3), (a_2, h_4)$	$(a_1, h_3), (a_2, h_4)$
3	h_5	$\varnothing$	$\varnothing$	(a_1, h_5)	(a_1, h_5)

Table 1: A. Elicited Demand Of Households at step 1

The institutions that demand each apartment are $I^1_{a_1} = \{1, 2, 3\}$, $I^1_{a_2} = \{1, 2\}$ and $I^1_{a_3} = \{1\}$. The NDA algorithm then moves to Phase B. The iterations over the sets M^1_i are shown in Table 2. Finally, the last column of this table illustrates the tentative assignment produced at the end of step 1.

I	$\tilde{M}^1_i$	$Ch_i(\tilde{M}^1_i, q_i)$	$\mu^1(i)$
1	(a_3, h_6)	(a_3, h_6)	(a_3, h_6)
2	$(a_1, h_3), (a_2, h_4)$	$(a_1, h_3), (a_2, h_4)$	$(a_1, h_3), (a_2, h_4)$
3	$\varnothing$	$\varnothing$	$\varnothing$

Table 2: B. Iteration over M^1_i at step 1

Table 3 shows the demands of households at step 2. The set of institutions that demand each apartment are $I^2_{a_1} = \{1, 2\}$, $I^2_{a_2} = \{1, 2, 3\}$ and $I^2_{a_3} = \{1\}$. Phase B of the NDA algorithm results in the sets M^2_i which are summarized in Table 4, where the last column shows the tentative assignment produced at the end of step 2. From Table 4, we observe that institution 2 is rejected from the apartment a_2 because it has a lower priority than institution 3 under the priority π_{a_2}. Moreover, households h_1, h_4, and h_6 have not yet been rejected by all their acceptable apartments. So, the algorithm goes to step 3.

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$	$M^2_i = \tilde{M}^0_i$	Ch^2_i
1	h_2	h_1	h_6	$(a_1, h_2), (a_2, h_1), (a_3, h_6)$	$(a_1, h_2), (a_2, h_1), (a_3, h_6)$
2	h_3	h_4	$\varnothing$	$(a_1, h_3), (a_2, h_4)$	$(a_1, h_3), (a_2, h_4)$
3	$\varnothing$	h_5	$\varnothing$	(a_2, h_5)	(a_2, h_5)

Table 3: A. Elicited Demand Of Households at step 2

I	$\tilde{M}^2_i$	$Ch_i(\tilde{M}^2_i, q_i)$	$\mu^1(i)$
1	(a_3, h_6)	(a_3, h_6)	(a_3, h_6)
2	(a_1, h_3)	(a_1, h_3)	(a_1, h_3)
3	(a_2, h_5)	(a_2, h_5)	(a_2, h_5)

Table 4: B. Iteration over M^2_i at step 2

Table 5 shows the demands of households during Phase A of step 3. Te sets of institutions that demand each apartment are given by $I^3_{a_1} = \{2\}$, $I^3_{a_2} = \{3\}$ and $I^3_{a_3} = \{1\}$, and Phase B starts. Table 6 shows the iterations over the sets M^3_i, where the last column shows the tentative assignment produced at the end of step 3.

I	$H^i_{a_1,i}$	$H^i_{a_2,i}$	$H^i_{a_3,i}$	$M^3_i = \tilde{M}^0_i$	Ch^3_i
1	$\varnothing$	$\varnothing$	h_1, h_2, h_6	(a_3, h_6)	(a_3, h_6)
2	h_3, h_4	$\varnothing$	$\varnothing$	$(a_1, h_3), (a_1, h_4)$	(a_1, h_3)
3	$\varnothing$	h_5	$\varnothing$	(a_2, h_5)	(a_2, h_5)

Table 5: A. Elicited Demand Of Households at step 3

I	$\tilde{M}^2_i$	$Ch_i(\tilde{M}^2_i, q_i)$	$\mu^1(i)$
1	(a_3, h_6)	(a_3, h_6)	(a_3, h_6)
2	(a_1, h_3)	(a_1, h_3)	(a_1, h_3)
3	(a_2, h_5)	(a_2, h_5)	(a_2, h_5)

Table 6: B. Iteration over M^3_i at step 3

Note that household h_4 is the unique household that has not yet been rejected by all her acceptable apartments. Her last acceptable apartment is a_3. The algorithm moves to step 4.

At step 4, household h_4 demands the apartment a_3. Household h_6 also demands this apartment because $(a_3, h_6) \in \theta^3(1)$, see Table 6. So, the set of institutions that demand a_3 is $I^4_{a_3} = \{1, 3\}$, where institution 1 has a higher priority than institution 3 under π_{a_3}. Given that $\tau(h_4) = 3$, we conclude that household h_4 is rejected from apartment a_3. The NDA algorithm stops and the final assignment is

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ \varnothing & \varnothing & a_1 & \varnothing & a_2 & a_3 \\ \varnothing & \varnothing & 2 & \varnothing & 3 & 1 \end{pmatrix}.$$

□

The previous example shows the importance of the iteration over the sets M^t_i. Without this stage, the pair $(h_5, 3)$ would have justified envy over the pair $(h_4, 2)$ at the apartment a_2.

3.2 No distributional constraints

In this section we analyze the NDA algorithm when that there are no distributional constraints – quotas are ineffective. (This situation arises when institutions' quotas are large enough i.e. $q_i > A$ for all $i \in I$.)The implications for the NDA mechanism are presented below.

We first show that in a market without distributional constraints, the "inner loop" of the NDA algorithm (phase B) only requires one step.

Lemma 3.1. *Consider a matching market with institutions and no distributional constraints, i.e. $q_i > \#A$ for all $i \in I$. Then, phase B of the NDA algorithm is iterated only once.*

When there are no distributional constraints, we also show that all apartments are assigned at the end of the procedure.

Lemma 3.2. *Consider a matching market with institutions and no distributional constraints, i.e. $q_i > \#A$ for all $i \in I$. If an apartment is assigned at some step t by some institution, this apartment is assigned under the assignment μ^{NDA}.*

The following theorem shows all the desirable properties that the NDA algorithm satisfies when there are no distributional constraints.

Theorem 3.1. *Consider a matching market with institutions* $(I, Q, H, \tau, A, P, \succ, \pi)$ *where* $q_i > \#A$ *for all* $i \in I$.

1. *The* μ^{NDA} *assignment is individually rational, non-wasteful and there is no justified envy; namely, the assignment* μ^{NDA} *is fair.*

2. *There is no fair assignment that Pareto dominates* μ^{NDA}.

3. *The NDA mechanism is strategy-proof for households.*

Theorem 3.1 shows that in the absence of distributional constraints, the model of matching through institutions inherits the properties of classical school choice problems. If, in the final matching, institutions do not claim a fixed quota of assignments, their presence does not fundamentally change the assignment mechanism. Even though the description of assignments is more complex in matching through institutions than in classical school choice problems, the properties of the assignment rule does not differ significantly. One difference, emphasized in the following example, is that institutions can benefit from misrepresenting their preferences, so that the NDA is not strategy-proof for institutions.

Example 3.2. Consider a market such that $I = \{1, 2\}$, $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ and $A = \{a_1, a_2, a_3, a_4\}$. The vector of quotas is $Q = (3, 1)$, and the type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_3, h_4\}$ and $\tau^{-1}(2) = \{h_5, h_6, h_7\}$. The institutions priorities, households preferences and apartments priorities are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_3, h_5) \\ (a_2, h_2) & (a_1, h_6) \\ (a_3, h_3) & (a_1, h_5) \\ (a_4, h_4) & (a_3, h_7) \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} & P_{h_6} & P_{h_7} \\ a_1 & a_2 & a_3 & a_4 & a_1 & a_1 & a_4 \\ a_2 & a_3 & a_4 & a_3 & a_2 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_1 & a_3 & a_3 & a_1 \\ a_4 & a_1 & a_2 & a_2 & a_1 & a_4 & a_2 \end{pmatrix} \text{ and}$$

$$\pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} & \pi_{a_4} \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

Running the NDA algorithm, we get the following assignment

$$\mu^{NDA}[\succ] = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ \varnothing & a_2 & a_3 & a_4 & \varnothing & a_1 & \varnothing \\ \varnothing & 1 & 1 & 1 & \varnothing & 2 & \varnothing \end{pmatrix}.$$

Now, consider that institution 1 has the following preference

$$\succ^{1'} = \begin{pmatrix} (a_1, h_1) \\ (a_2, h_2) \\ (a_4, h_4) \\ (a_3, h_3) \end{pmatrix}.$$

Running the NDA with priorities $\succ' = (\succ^{1'}, \succ^2)$, we get that

$$\mu^{NDA}[\succ'] = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_2 & \varnothing & a_4 & a_3 & \varnothing & \varnothing \\ 1 & 1 & \varnothing & 1 & 2 & \varnothing & \varnothing \end{pmatrix}.$$

If institution 1 prefers $\{(a_1, h_1), (a_2, h_2), (a_4, h_4)\}$ to $\{(a_2, h_2), (a_3, h_3), (a_4, h_4)\}$, because (a_1, h_1) is preferred to any other acceptable pair (a, h), then the institution 1 can improve its final allocation by misreporting its priorities. Note that such assumption does not contradict the fact that preferences $\succ^i$ are responsive.

4 Distributional constraints

In a model with distributional constraints, quotas are effective and each institution must fill its quota. Formally, consider a market with institutions $(I, Q, H, \tau, A, P, \succ, \pi)$ such that $\sum_{i=1}^{N} q_i = A$. An assignment μ **respects the distributional constraints** if $|\theta(i)| = q_i$ for all institutions $i \in I$.

It is important to note that an assignment that satisfies the distributional constraints does not always exist.

Example 4.1. Let $I = \{1, 2\}$, $\{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4\}$, where households type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_3\}$ and $\tau^{-1}(2) = \{h_4\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1,h_1) & (a_3,h_4) \\ (a_3,h_2) & \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} \\ a_1 & a_1 & a_1 & a_1 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_2 & a_3 \end{pmatrix}.$$

Since no acceptable pair (a,h) for 1 and 2 include the apartment a_2, there is no IR assignment that satisfies the distributional constraints. □

From Example 4.1, we note that assuming that the number of households attached to i is greater than the quota ($|H_i| \geq q_i$), is not sufficient to guarantee the existence of an assignment that satisfies the distributional constraints. Even when an assignment that respects distributional constraints exists, the NDA does not always choose it, as the following example shows:

Example 4.2. (**No distributional constraints with single types**). Consider $I = \{1,2\}$, $H = \{h_1,h_2,h_3\}$ and $A = \{a_1,a_2,a_3\}$. The vector of quotas is $Q = (q^1,q^2) = (2,1)$, and the type function is given by $\tau^{-1}(1) = \{h_1,h_2\}$ and $\tau^{-1}(2) = \{h_3\}$. The profiles of institutions priorities, households preference and apartments priorities are

$$\begin{pmatrix} \succ^1 & \succ^2 \\ (a_2,h_2) & (a_1,h_3) \\ (a_1,h_1) & \\ (a_1,h_2) & \\ (a_3,h_1) & \end{pmatrix}, \quad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_1 \\ a_2 & a_2 & a_3 \\ a_3 & a_3 & a_2 \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Running the NDA, we have that $H^1_{a_1,1} = \{h_1,h_2\}$ and $H^1_{a_1,2} = \{h_3\}$. Given that $(a_1,h_1) \succ^1 (a_1,h_2)$ and $1\pi_{a_1}2$, the tentative assignment produced at then of the step 1 is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & \varnothing & \varnothing \\ 1 & \varnothing & \varnothing \end{pmatrix}.$$

Then, during the step 2 we have that $H^2_{a_1,1} = \{h_1\}$, $H_{a_2} = \{h_2\}$ and $H^2_{a_3,2} = \{h_3\}$. Consequently, we get

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \varnothing \\ 1 & 1 & \varnothing \end{pmatrix}.$$

At step 3, households h_3 points to her last acceptable apartment a_2. However, (a_2, h_3) is not acceptable for institution 2. Then, h_3 is rejected from a_3 and the algorithm stops. The final assignment is $\mu^{NDA} = \mu^2$, which does not satisfy the distributional constraints.

Moreover, there exists an assignment that satisfies the distributional constraints.

$$\mu' = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_3 & a_2 & a_1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Therefore, the market does not satisfy the distributional constraints because no acceptable pair, for each institution, remains unassigned under the assignment μ'. □

On the contrary, when demand is sufficiently high – as is surely the case for social housing in Paris – distributional constraints can be satisfied and the NDA outputs an assignment satisfying the constraints. We formalize this intuition by stating the following **over-demand condition**:

Assumption 1. There is at least one IR assignment that satisfies the distributional constraints, and for all such IR assignments, for all institutions i there is an unassigned household h and an apartment a such that $(a, h) \succ^i \varnothing$ and $aP_h\varnothing$.

The presence of distributional constraints may prevent the NDA from outputting fair outcomes. During Phase B of the NDA algorithm, institutions make offers following the preferences of households attached to them and they might temporarily fill their quotas. Later in the run of the NDA, better options might arise for some institutions, leading them to drop some apartments. In this case, institutions act as interrupters, as defined in Kesten (2010): they temporarily accept pairs of households and apartments which will be dropped in the final outcome, preventing the emergence of fair outcomes. The following example shows that the NDA may produce an assignment which fails to satisfy fairness over households of the same type.

4.1 Interrupters

Example 4.3. (There is justified envy for households of the same type). Let $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, \ldots, h_7\}$, where households type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_5, h_6\}$ and $\tau^{-1}(2) = \{h_3, h_4, h_7\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1,h_1) & (a_2,h_3) \\ (a_1,h_2) & (a_2,h_4) \\ (a_2,h_1) & (a_1,h_3) \\ (a_2,h_2) & (a_1,h_4) \\ (a_1,h_5) & (a_1,h_7) \\ (a_2,h_6) & (a_2,h_7) \\ (a_3,h_1) & \\ (a_3,h_6) & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} & P_{h_6} & P_{h_7} \\ a_1 & a_2 & a_1 & a_1 & a_2 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_2 & a_1 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \end{pmatrix}.$$

The profile of apartment priorities is $\pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Running the NDA algorithm, Phase A of step 1 is described in Table 7. Observe that apartment a_1 is demanded by both institutions, where $2\pi_{a_1}1$. As a consequence, Phase B stops in one step. So, the tentative assignment produced at the end of step 1 is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ \varnothing & a_2 & a_1 & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^1_{a_1,i}$	$H^1_{a_2,i}$	$H^1_{a_3,i}$	$M^1_i = \tilde{M}^0_i$	Ch^1_i
1	h_1, h_6	h_2, h_5	$\varnothing$	$(a_1,h_1),(a_2,h_2)$	$(a_1,h_1),(a_2,h_2)$
2	h_3, h_4, h_7	$\varnothing$	$\varnothing$	$(a_1,h_3),(a_1,h_4),(a_1,h_7)$	(a_1,h_3)

Table 7: A. Elicited Demand Of Households step 1

The algorithm continues to step 2, the elicited demand of households at this step is summarized in Table 8. Again, institutions 1 and 2 demand the apartments a_1 and a_2. The sets of institutions that demand each apartment are $I^2_{a_1} = \{1, 2, \} = I^1_{a_2}$ where institution 2 has a higher priority than institution 1 under priorities π_{a_1} and π_{a_2}. Moreover, we have that $q_2 = 1$ and $(a_2, h_4) \succ^i (a, h)$

for all $(a,h) \in M_2^2$. So, the tentative assignment produced at the end of step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ \varnothing & \varnothing & \varnothing & a_2 & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & 2 & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$	$M_i^2 = \tilde{M}_i^0$	Ch_i^2
1	h_5	h_2, h_6, h_1	$\varnothing$	$(a_2,h_1),(a_2,h_2)$	(a_2,h_1)
2	h_3	h_4, h_7	$\varnothing$	$(a_1,h_3),(a_2,h_4)$	(a_2,h_4)

Table 8: A. Elicited Demand Of Households step 2

It is important to note that the assignments $(h_3, a_1, 2)$ and $(h_2, a_2, 1)$ are disrupted at the end of step 2 because $2\pi_{a_1}1$ and $(a_2, h_4) \succ^2 (a_2, h_2)$. Households h_2 and h_3 were rejected from apartments a_2 and a_1, respectively. The algorithm continues to step 3 because not all households have been rejected by all their acceptable apartments. For example households h_2 and h_3 have not been rejected from the apartments a_1 and a_2, respectively. Table 9 shows the Phase A of step 3.

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$	$M_i^3 = \tilde{M}_i^0$	Ch_i^3
1	h_2	$\varnothing$	h_1, h_5, h_6	$(a_1,h_2),(a_3,h_1),(a_3,h_6)$	$(a_1,h_2),(a_3,h_1)$
2	$\varnothing$	h_3, h_4	h_7	$(a_2,h_3),(a_2,h_4),(a_3,h_7)$	(a_2,h_3)

Table 9: A. Elicited Demand Of Households step 3

According to Table 9, the apartments demanded by each institution are different. Then, the tentative assignment produced at the end of step 3 is

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

In the assignment μ^3 we note that household h_4 has been rejected from apartment a_2. Also, households h_5, h_6 and h_7 have been rejected from all their acceptable apartments.

The algorithm continues to step 4, where household h_4 demands the apartment a_3, her last acceptable apartment, and households h_1, h_2, h_3 iterate their demand to their match. We know that institution 1 has a higher priority than institution 2 under the priority π_{a_3}, which implies that household h_4 is rejected from the apartment a_3 because $1 \notin 2$ and $\tau(h_4) = 2$. The NDA algorithm stops, and the final assignment is

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

We now observe that this assignment does not satisfy fairness over households of the same type. Considering institution 1, households h_1 and h_2, we know that

$$a_1 P_{h_1} a_3 \qquad \text{and} \qquad (a_1, h_1) \succ^1 (a_1, h_2),$$

where $\varphi_I^{NDA}(h_1) = \varphi_I^{NDA}(h_2) = 1$, $\varphi_A^{NDA}(h_2) = a_1$ and $\varphi_A^{NDA}(h_1) = a_3$. The pair $(h_1, 1)$ has justified envy over $(h_2, 1)$ at apartment a_1.

We also note that the sets $\{(a_1, h_5), (a_2, h_6), (a_3, h_3)\}$ and $\{(a_1, h_4), (a_2, h_4), (a_1, h_7), (a_2, h_7)\}$ are sets of unassigned pairs that are acceptable for institutions 1 and 2, respectively. [3] $\square$

Example 4.3 illustrates how an interruption works. Observe that institution 2 is tentatively assigned to a_1 at step 1. Thus, household h_1 and institution 1 are displaced from a_1 at the end of step 1. Similarly, household h_3 and institution 2 are displaced from apartment a_1 at step 2 (see Table 8.) However, apartment a_1 is assigned to institution 1 at step 3. But household h_1 can no longer demand this apartment because she was rejected from it at step 1. As a consequence, the choice function of institution 1 does not consider the pair (a_1, h_1), which implies the presence of justified envy between the pairs $(h_1, 1)$ and $(h_2, 1)$. Following Kesten (2010)'s terminology, we say that institution 2 is an interrupter for apartment a_1. We formally define an interrupter below.

Given a matching problem to which the NDA is applied, we say that an institution i is an **interrupter for** apartment a if there exists

1. steps t to $t + n$ such that $a \in \theta^{t'}(i)$ for all $t' \in \{t, t+1, \ldots, t+n\}$ but $a \notin \theta^{t'}(i)$ for all $t' > t + n$, and

2. an institution $j \neq i$ and a household h such that $(a, h) \in Ch_j(M_j^{t'}, q_j)$ but $(a, h) \notin \mu^{t'}(j)$ for some $t' \in \{t, t+1, \ldots, t+n\}$.

[3] Since μ^{NDA} satisfies the distributional constraints, the over-demand condition holds in this example.

4.2 Nested Deferred Acceptance with Interrupters

Following the Efficiency Adjusted Deferred Acceptance Mechanism (Kesten, 2010), we modify the NDA introducing a second stage where we search for all interrupter institutions. Then we let these institutions delete from their preference the pairs containing the apartment where they cause the interruption. We define the following delete operation on priorities $\succ^i$.

Let $\Im$ be the set of all possible priorities $\succ^i$, for all $i \in I$. The **delete operation** over $\Im$ is the function $\setminus : \Im \times (A \times H) \to \Im$ such that $\setminus(\succ, a)$, or simply $\succ \setminus a$, is the preference that declares all pairs $(a, h) \succ^i \varnothing$ as unacceptable for i. In other words, the preference $\succ \setminus a$ drops all acceptable pairs (a, h) from $\succ^i$ and holds the original order in the preference $\succ^i$. Note that Kesten (2010) defines this operation over students' preferences, (the equivalent in our model of households in the outer loop), because he identifies that students causes the loss of efficiency during the Deferred Acceptance algorithm. In our case, the delete operation targets institutions, which are involved in the inner loop of the algorithm, as they are the source of the loss of fairness of the NDA algorithm.

We now finally introduce the Nested Deferred Acceptance with Interrupters (NDAI). Each step of this mechanism has two stages: the NDA algorithm runs in the first stage, while the second stage deletes pairs from the priorities of interrupters. Formally, the NDAI proceeds as follows.

Initialization

Initialize the counter of iterations over interrupter institutions at $x := 0$.

Step 0. This step is divided in the following stages:

Stage 0.1 NDA Phase. Let $\succ^0 = (\succ^{i,0})_{i \in I} = (\succ^i)_{i \in I}$. Run the NDA algorithm using the profile of priorities and preferences $(\succ^0, P)$.

Stage 0.2 Deletion in Priorities If there is no interrupter, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{i,1} = \succ^{i,0} \setminus a$; $\succ^{j,1} = \succ^{j,0}$ if j is not an interrupter.

Step x. The stages are the following.

Stage x.1 NDA Phase. Run the NDA algorithm with the profile of priorities and preferences $(\succ^x, P)$.

Stage x.2 Deletion in Priorities. If there is no interrupter, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{ix+1}=\succ^{i,x} \setminus a$; $\succ^{j,(x+1)}=\succ^{j,x}$ if j is not an interrupter.

The output of the previous mechanism is denoted by $\mu^{NDAI}[H, A, P, I, \succ, \pi_A, q]$, or simply μ^{NDA} whenever there is no confusion. The NDAI is solvable in polynomial time because each NDA phase is solvable in a finite number of steps, and there are at most $|I|$ interrupters. The following example illustrates how the NDAI algorithm works.

Example 4.4. We consider the same market as in the Example 4.3 and consider stage 0.1 of the NDAI algorithm. We observe that institution 2 causes an interruption over the pair (a_1, h_1). Consequently, we delete all the pairs (a_1, h) from the preference $\succ^2$ at Stage 0.2. We get

$$\succ^{1,1}=\succ^1 \text{ and } \succ^{2,1}= \begin{bmatrix} (a_2, h_3) \\ (a_2, h_4) \\ (a_2, h_7) \end{bmatrix}.$$

I	$H^{i1}_{a_1}$	$H^{i1}_{a_2}$	$H^{i1}_{a_3}$
1	h_1, h_6	h_2, h_5	$\varnothing$
2	$h_3,\ h_4, h_7$	$\varnothing$	$\varnothing$

Table 10: A. Elicited Demand Of Households step 1, Stage 1.1

Consider stage 1.1. We run the NDA algorithm with priorities $\succ^1$. Step 1.1 of this NDA algorithm is summarized in Table 10. Given that $q_1 = 2$, and $1\pi_{a_1}2$, the tentative assignment is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

The NDA in Stage 1.1 moves to step 2. Phase A of the algorithm is illustrated in the Table 11. Following the institutions priorities and the fact that $2\pi_{a_3}1$, the tentative assignment produced at the end of step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$
1	h_1, h_5	h_2, h_6	$\varnothing$
2	$\varnothing$	h_3, h_4, h_7	$\varnothing$

Table 11: A. Elicited Demand Of Households step 2

Now, the NDA algorithm of Stage 1.1 moves to step 3. We show the demands of households in Table 12. Since the institution 1 has a higher priority than institution 2 under π_{a_3}, we get the tentative assignment μ^3.

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & a_3 & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & 1 & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$
1	h_1, h_2	$\varnothing$	h_5, h_6
2	$\varnothing$	h_3	h_4, h_7

Table 12: A. Elicited Demand Of Households step 3

Note that household h_2 has no been rejected by the apartment a_3, her last acceptable apartment. The NDA at Stage 1.1 moves to step 4 where Table 13 shows Phase A.

We observe that all households have been accepted or rejected at the end of the step 4, and hence Stage 1.1 stops. There are no interrupters because no apartment is rejected by any institution. Therefore, the NDAI algorithm stops and produces the following assignment.

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^4_{a_1,i}$	$H^4_{a_2,i}$	$H^4_{a_3,i}$
1	h_1	$\varnothing$	h_2, h_6
2	$\varnothing$	h_3	$\varnothing$

Table 13: A. Elicited Demand Of Households step 4

This assignment satisfies fairness over households of the same type.

□

The following theorem summarizes all the desirable properties that the NDAI algorithm satisfies in an over-demanded market.

Theorem 4.1. *Consider a matching market with institutions* $E = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$ *where the over-demand condition is satisfied for all institutions.*

1. *The* μ^{NDAI} *is individually rational, non-wasteful, respects distributional constraints and there is no justified envy; namely, the assignment* μ^{NDAI} *is fair.*
2. *There is no assignment which is fair over households of the same type that Pareto dominates* μ^{NDAI}.
3. *The NDAI is strategy-proof for households.*

Theorem 4.1 extends the classical properties of the deferred acceptance algorithm to the model of matching through institutions with distributional constraints. We observe that the induction hypothesis used in the proof that the assignments produced by the NDA are not Pareto dominated by other fair assignments in Theorem 3.1 does not hold under distributional constraints. In fact, assignments do not have a classical lattice structure. As the following example shows, households may disagree on which is the best fair assignment satisfying distributional constraints.

Example 4.5. Consider a market such that $H = \{h_1, h_2\}$, $A = \{a_1, a_2\}$ and $I = \{1, 2\}$. Institutions priorities, households preferences and apartments priorities are given by.

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_2, h_2) \\ (a_2, h_1) & (a_1, h_2) \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & a_2 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ i_2 & i_1 \\ i_1 & i_2 \end{pmatrix}.$$

In this market, we have the following fair assignments where the distributional constraints hold.

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ i_1 & i_2 \end{pmatrix} \text{ and } \mu^{NDAI} = \begin{pmatrix} h_2 & h_1 \\ a_1 & a_2 \\ i_2 & i_1 \end{pmatrix}.$$

We note that h_1 prefers μ^{NDAI} to μ, while h_2 has reverse preferences.

4.3 Multiple institutions

In this subsection we generalize our previous results by allowing households to be attached to multiple institutions, as is the case for applications to social housing in Paris. We replace the type assignment mapping τ by the following multiple institutions assignment $\tau : H \to 2^I$, i.e $\tau(h) \subseteq I$ and $H_i = \{h \in H \mid i \in \tau(h)\}$.

As we show in Appendix B, most of our results can be extended to the case of agents attached to multiple institutions, and Theorem 4.1 can be modified to apply to this situation. Interrupters can be defined in exactly the same way as in the case of agents attached to single institutions, and the modified NDAI produces individually rational and fair outcomes which are not Pareto dominated by any other fair assignment and is strategy-proof for households.

5 Distributional constraints and regional preferences

In this section we show how the Nested Deferred Acceptance algorithm can be applied to the markets with distributional constraints and regional preferences introduced by Kamada and Kojima (2014). This model is inspired by the assignment of doctors to hospitals in Japan. Instead of considering a simple many-to-one matching between hospitals and residents, Kamada and Kojima (2014) note that there exists some flexibility in the way hospitals fill their quotas of positions, and introduce "regional preferences" over the pairs of hospitals and doctors.

A market with distributional constraints and regional preferences is defined by:

1. $D = \{d_1, d_2, \ldots, d_D\}$ is the finite set of doctors, a generic doctor is denoted by d;

2. $H = \{h_1, h_2, \ldots, h_H\}$ is the finite set of hospitals, a generic hospital is denoted by h;

3. $Q = (q_{h_1}, q_{h_2}, \ldots, q_{h_H})$ is the vectors of quotas, where q_h is the quota of the hospital h, a generic quota is q;

4. $R = \{1, 2, \ldots, R\}$ is the finite set of regions, a generic region is r;

5. $\tau : H \to R$ is the region function, i.e. if a hospital h belongs to the region r, we write that $\tau(h) = r$. Let H_r be the set of hospitals in region r, note that $H_r \cap H_{r'} = \varnothing$ for region $r' \neq r$;

6. $P = (P_{d_1}, P_{d_2}, \ldots, P_{d_D})$ is the vector of doctors' preferences, P_d is the strict preference of household $h \in H$ over $H \cup \varnothing$; hP_dh' means that doctor d prefers h to h', a hospital h is acceptable for doctor d if $hP_h\varnothing$.

7. $\succ = (\succ_{h_1}, \succ_{h_2}, \ldots, \succ_{h_H})$ is the profile of hospitals priorities over the set of doctors D. We assume that for each $h \in H$ the preference $\succ_h$ is responsive on 2^D, i.e. for any $d, d' \in D$ and $S \in 2^D$ we have that

   i. $S \cup \{d\} \succ_h S \cup \{d'\}$ if and only if $d \succ_h d'$, and

   ii. $S \cup \{d\} \succ_h S$ if and only if $d \succ_h \varnothing$;

8. $\succsim_r$ is the regional preference of r over the set of vectors $W_r = \{w = (w_h)_{h \in H} | w_h \in \mathbb{Z}_+\}$, where w_h specifies the number of doctors allocated to each hospital h in region r;

9. There is a vector of regional caps $\tilde{Q} = (q_r)_{r \in R}$, where q_r is a non-negative integer for each region r.

We say that $\tilde{E} = (D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ is a **market with distributional constraints and regional preferences**. It is important to note that this market shares some common features with a matching market through institutions. Since doctors are interested in hospitals, and regions care about the number of doctors that each hospital can accept, we have that doctors, regions and hospitals play a similar role to households, institutions and apartments, respectively.

The main differences between Kamada and Kojima (2014) and our model are related first with how preferences of institutions on regions are defined. In order to satisfy a regional cap, regions priorities are defined over the set of capacity vectors. That is to say, regions do not care about specific hospitals, they only care about the number of doctors that each hospital is willing to accept. Thus, Kamada and Kojima introduced quasi-choice rules in order to pick the prefered capacity vector given the regional cap. Given $\succsim_r$, a function $\tilde{Ch}_r : W_r \times \mathbb{Z}_+ \to W_r$ is an **associated quasi choice rule** if $\tilde{Ch}_r(W_r, q_r) \in argmax_{\succeq_r}\{w \in W_r | \#w \leq q_r\}$ for any non-negative $w = (w_h)_{h \in H_r}$.

Moreover, each hospital belongs to one and only one region. This means that the type function is defined over the goods to be consumed. On the opposite, a type function in a matching market through institutions is defined over the set of agents interested in being assigned to some good, namely households.

We adapt the NDA algorithm 1) by selecting unmatched doctors sequentially instead of simultaneously to make offers and 2) by introducing the quasi choice of regions, instead of the choice of institutions.

Initialization

Consider a market $(D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ with distributional constraints and regional preferences. The matching is initialized to be the empty matching, so $\mu^0(h) = \mu^0(d) = \mu^0(r) = \varnothing$, for all $d \in D$, $h \in H$ and $r \in R$.

For all doctors $d \in D$, let $H_d^t := H$, and $t = 1$. For each region r, fix a quasi-choice rule $\tilde{Ch}_r$.

A. Eliciting the demand of doctors

Arbitrarily pick one unassigned doctor d, who asks for the most preferred hospital in H_d^t, denoted by D_d^t while r is the region of D_d^t; moreover matched doctors d' in region r iterate their demand to their match, $D_{d'}^t = \{\mu^{t-1}(d)\}$.

For all hospitals $h \in H$ in region r, we define the set of doctors that demand hospital h in region r as follows:

$$D_{h,r}^t = \{d \in D | D_d^t = \{h\}, d \succ_h \varnothing \text{ and } \tau(h) = r\}.$$

The set of pairs (d, h) that can be assigned to region r is defined as

$$M_r^t = \{(d, h) \in D \times H | d \succ_h \varnothing \text{ and } d \in D_{h,r}^t\}.$$

The possible assignments are

$$\mathcal{P}_r^t = \left\{p = \{(d, h)\}_{\tau(h)=r} \mid (d, h) \in M_r^t \text{ and } d \text{ is not matched twice}\right\}.$$

The number of doctors matched to hospital h at p is $w_h(p) = \#\{d \in D \mid (d,h) \in p\}$, thus, the set of capacity vectors is

$$W_r^t = \left\{ w = (w_h)_{h \in H} | \exists\ p \in \mathcal{P}_r^t \text{ and } w_h = w_h(p) \text{ for all } h \text{ in region } r \right\}.$$

B. Matching the demand of region r and hospitals of the region.

B.1 Regions r demands the vector

$$\omega_r^t = (\omega_h^t)_{\tau(h)=r} = \tilde{Ch}_r(W_r^t, q_r).$$

B.2 Each hospital h in region r is tentatively assigned to the preferred subset of $D_{h,r}^t$ with cardinality w_h^t.

The assignment in other regions remains the same.

C. Iteration over $\mathbf{D_d^t}$

Let $H_d^{t+1} := H_d^t \setminus \{\max_{P_d} H_d^t\}$, $t := t+1$.

If all doctors have been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the outcome assignment. Otherwise, go to the Phase A.

The assignment produced by the previous algorithm is denoted by $\tilde{\mu}^{NDA}$. It depends on a market $\tilde{E}$ and a fixed associated quasi choice rule $\tilde{Ch}$.

In order to find an assignment between hospitals and doctors that respect the distributional constraints and regional caps, Kamada and Kojima introduced the Generalized Flexible Deferred Acceptance (GFDA) algorithm.[4]

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[4]For each region r fix a quasi-choice rule $\tilde{Ch}_r$. The GFDA algorithm proceed as follows

1. Begin with an empty matching, i.e. $\mu_d = \varnothing$ for all $d \in D$.
2. Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
3. Let d apply to the most preferred hospital $\overline{h}$ at H_d among the hospitals that have not rejected d so far. If d is unacceptable to $\overline{h}$, then reject this doctor and go back to step 2. Otherwise, let r be the region such that $\overline{h} \in H_r$ and define vector $\omega = (\omega_h)_h \in H_r$ by

   (a) $\omega_{\overline{h}}$ is the number of doctors currently held at $\overline{h}$ plus one, and

   (b) w_h is the number of doctors currently held at h if $h \neq \overline{h}$,

Theorem 5.1. *Consider a market with distributional constraints, then the matchings generated by NDA and GFDA are the same when the same unmatched doctor is selected to make offer at each iteration.*

Proof. Let $\tilde{E}$ be a fix market with distributional constraints and regional preferences. We proceed by induction over iterations to prove that tentative matchings are the same when the same unmatched doctor is selected to make offer.

Hypothesis of induction. We assume that the tentative matching generated by NDA and GFDA are the same, as well as the set to which households did not make and offer, at iteration t when the same unmatched doctor is selected to make offer.

Induction step. Consider that doctor d is selected by both algorithms to make offer at step $t+1$. In both algorithms, d has not been rejected by all her acceptable hospitals, otherwise both algorithms choose another doctor.

According to the GFDA, doctor d applies to her most preferred hospital h at H_d^t and doctors iterate their demand during the step $t+1$. Consequently, at the end of GFDA's phase 2, doctors have elicited their demand. By the hypothesis of induction, tentative matchings generated at iteration t are the same, then the demand elicited at GFDA's phase 2 is the same as the demand elicited by doctors at the end of NDA's phase A.

At phase 3 of GFDA, if the new application is acceptable for h, it increases the number of doctors held at h by one. This implies that the pair (d, h) is an acceptable pair for the region r. Similarly, the pair belongs to the set M_r^{t+1} of NDA's Phase A.

Finally, during the GFDA's phase 4, the quasi choice rule determines the number of doctors that each hospital can hold. As a consequence, hospitals holds their $(\tilde{Ch}_r(w))_h$ most preferred doctors and rejects the rest, posibly including the applicant h. Given that W_r^{t+1} is generated by the sets M_r^{t+1}, during the NDA's phase B the vector of capacities chosen by the quasi-choice rule is the same as in the GFDAs phase 4. Therefore, the tentative matching generated at $t+1$ is the same for the GFDA and the NDA.

□

4. Each hospital $h \in H_r$ considers the new applicant d (if $h = \overline{h}$) and doctors who are temporarily held from the previous step together. It holds its $(\tilde{Ch}_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to step 2.

6 Concluding Remarks

We model a matching market with institutions as a three sided market. Agents have preferences over objects. Institutions have a list of agents and own objects and have preferences over assignments of agents to objects. Objects have priorities over institutions. We introduce a new assignment procedure, the Nested Deferred Acceptance (NDA) algorithm. If institutions do not face distributional constraints, the NDA is fair, Pareto undominated by any fair mechanism and strategy-proof. We analyse the consequences of imposing distributional constraints. We first show that an assignment respecting the constraints does not always exist. Second, we show that the NDA algorithm does not necessarily output fair matchings. We introduce an over-demand condition to guarantee the existence of assignments respecting distributional constraints, and we identify interrupters who are the origin of the lack of fairness, as in Kesten (2010), We introduce a new algorithm – NDA with interrupters – which preserves the properties of the NDA algorithm under the over-demand condition.

The model of matching through institutions we consider is inspired by the assignment of social housing in Paris but the procedure we propose applies more generally to situations where agents belong to different groups, and pool their resources to obtain more flexible outcomes. For example, we could apply the procedure to exchanges of students across universities, exchange of pupils across school districts, exchange of agents across regional waiting lists for organ transplant or social housing. Each of these assignment problems may have specific institutional constraints, and we hope to study them in detail in future work.

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A Proofs

Proof of Lemma 3.1:

Consider a step t of the NDA algorithm. We know that phase B starts with the set of acceptable apartment-households pairs $\tilde{M}_i^1 = M_i^t$ for all institutions i. This implies that each institution i demands the set $Ch_i(\tilde{M}_i^t, q_i)$, a set where apartments and households are not paired twice.

Let $A_i^t = \{a \in A | (a,h) \in M_i^t\}$ be the set of apartments demanded by some household of type i at step t. Since $|A_i^t| \leq A$, there are no distributional constraints and priorities are responsive, all acceptable pairs (a,h) belong to the set $Ch_i(\tilde{M}_i^1, q_i)$. Each institution i demands all the apartments in the set A_i^t. So, all the apartments in $\bigcup_{i \in I} A_i^t$ belong to $\theta^1(j)$, for some $j \in I$, at the end of Phase B.2.

Consequently

$$\tilde{M}_i^2 = \tilde{M}_i^1 \setminus \{(a,h) \in \tilde{M}_i^1 | a \in \theta^1(j) \text{ for some } j \neq i\} = \tilde{M}_i^1 \setminus \tilde{M}_i^1 = \varnothing$$

for all institution i. Therefore, phase B stops in one iteration.

Proof of Lemma 3.2: Consider an apartment a that is assigned by some institution i at some step t, i.e. a belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t+1$. Since there are no distributional constraints and priorities are responsive, the apartment a is assigned to some institution at the end of step $t+1$ (the institution in I_a^{t+1} with the highest priority at π_a). Iterating this argument, we conclude that the apartment a is assigned to some institution at all steps $t' \geq t$.

Therefore $\mu^{NDA}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps.

Proof of Theorem 3.1:

Individual Rationality. For all institutions $i \in I$, we know that $\mu^{NDA}(i) \subseteq Ch_i(M_i^T, q_i)$, where T is the last iteration of the NDA algorithm. Thus, $(a,h) \succ^i \varnothing$ for all $(a,h) \in \mu^{NDA}(i)$. Therefore, $\mu^{NDA}(i) \succ \varnothing$ for all $i \in I$.

Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

Non-wastefulness. We proceed by contradiction. We assume the existence of a household-institution pair (h,i) that claims an empty apartment a. That is to say, we have that i) $a P_h \varphi_A(h)$, ii) $(a,h) \in Ch_i(\mu^{NDA}(i) \cup (a,h), q_i)$, and iii) $\theta^{NDA}(a) = \varnothing$.

The condition i) means that household h demands the apartment a at some step t of the NDA algorithm. Moreover, condition ii) guarantees that the pair (a,h) is acceptable for the institution i. Thus, institution i demands the apartment a at step t. Applying Lemma 3.2, there exists some institution j such that $a \in \theta^{NDA}(j)$ which contradicts the condition iii).

There is no justified envy. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup (a, h), q_i)$,

iii. $i\pi_a i'$.

By condition i), household h demands the apartment a at some step t. Moreover, condition ii) ensures that the pair (a, h) is acceptable for the institution i, i.e. $(a, h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analyze the following cases.

Case I. $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDA}(h') = a$, household h' demands the apartment a at some step t'. Even more, $(a, h') \in Ch_i(M_i^{t'}, q_i)$ because (a, h') belongs to the set $\mu^{NDA}(i)$. Given that $(a, h) \in M_i^t$ but $\varphi_A^{NDA}(h) \neq a$, the definition of the choice function ensures that

$$(a, h') \succ^i (a, h). \tag{1}$$

Moreover, since no apartment can be paired twice, condition ii) implies that

$$(a, h) \succ^i (a, h'),$$

in contradiction with (1).

Case II. $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$. We know that $a \in \theta^{NDA}(i')$, which implies the existence of some step t' where

$$i'\pi_a j \text{ for all } j \in I_a^t, \text{ for all } t \geq t',$$

according to Phase B.3. In particular

$$i'\pi_a i,$$

in contradiction with condition iii).

In all cases we get a contradiction, therefore there is no justified envy at the assignment μ^{NDA}. So, this assignment is fair.

Pareto undominated. We proceed as in Gale and Shapley (1962). To prove that μ^{NDA} is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment a is said to be **achievable** for a household h if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu_A(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm.

Hypothesis of induction. At step t we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

Induction step. Consider that some household h^* is rejected at step $t+1$ from an apartment a. We assume, on the contrary, that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair (a, h^*) is acceptable for the institution i^*.

Now, let h be the household assigned to the apartment a at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

Case I. $i = i^*$. Since $\varphi^{t+1}(h) = (a, i)$, the apartment a belongs to $\theta^{t+1}(i)$. That is to say

$$(a, h) \succ^i (a, h^*) \tag{2}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that

$$(a, h) \in Ch_i(\mu(i) \cup (a, h)).$$

Note that h prefers a to all the apartments that have not rejected her, then the induction hypothesis ensures that household h prefers a to any other achievable apartment for her:

$$aP_h\varphi^\mu_A(h).$$

Moreover, $(a, h^*) \in \mu(i)$. That is to say, the pair (h, i) has justified envy over the pair (h^*, i) at the apartment a in the assignment μ, which contradicts the fact that μ is a fair assignment.

Case II. $i \neq i^*$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I^{t+1}_a$. Moreover, $i \in I^{t+1}_a$ because $\varphi^{t+1}(h) = (a, i)$. Given that $a \in \theta^{t+1}(i)$, we conclude that $a \notin \theta^{t+1}(i^*)$ because $i = \max_{\pi_a} I^{t+1}_a$; thus

$$i\pi_a i^*. \tag{3}$$

By induction hypothesis, we know that household h strictly prefers a to any other achievable apartment for her, i.e.

$$aP_h\varphi^\mu_A(h). \tag{4}$$

Now, we know that $(a,h) \in \mu^{t+1}(i)$, this means that the pair (a,h) is acceptable for institution i. Moreover, $(a,h^*) \in \theta^{\mu}(i^*)$, thus a is not assigned to i at μ. Also, there are no distributional constraints and institutions priorities are responsive, then

$$(a,h) \in Ch_i(\mu(i) \cup (a,h), q_i). \tag{5}$$

By 3, 4 and 5, the pair (h,i) has justified envy over the pair (h^*, i^*) at the apartment a in the assignment μ, which contradicts the fact that μ is fair.

In any case, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Truth-telling is a dominant strategy for households. We construct the proof as in Roth (1982).

For each household h, we say that P'_h is a **successful** misrepresentation of P if P'_h is a preference list such that

$$\varphi_A^{NDA}[P'_h, P_{-h}](h) P_h \varphi_A[P](h).$$

Let $a' := \varphi_A^{NDA}[P'_h, P_{-h}](h)$, we define the preference list P''_h where the apartment a' is declared as the most preferred apartment of h. Let P' and P'' be the preference profiles where household h reports P'_h and P''_h, respectively, and other households do not change their true preferences. The following lemma establishes that P'_h and P''_h are equivalents in the sense that

$$\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h).$$

Lemma A.1. *Consider a matching market through institutions with no distributional constraints. Then* $\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h)$.

Proof. By paragraphs above, the assignment $\mu^{NDA}[P']$ is fair. Let $\tau(h) := i$, it is no possible at $\varphi_A^{NDA}[P'_h, P_h]$ that (h,i) has justified envy over other pairs under P'' (because no apartment is preferred to a' under P''_h). This means that the assignment $\mu^{NDA}[P']$ is fair with respect to the profile P'', i.e. a' is achievable for h under the preference profile P''. Moreover, the apartment a' is the best acceptable achievable apartment of h under P'', and since $\mu^{NDA}[P'']$ is Pareto undominated by other fair assignments, we conclude that

$$\mu^{NDA}[P'](h) = \mu^{NDA}[P''](h).$$

□

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

Lemma A.2. *Consider P'_h a preference list different from the true preference list of h. If $\varphi_A^{NDA}[P'](h)$ is weakly preferred to $\varphi_A^{NDA}[P](h)$, then for each household $h' \neq h$, either*

$$\varphi_A^{NDA}[P'](h') P_{h'} \varphi_A^{NDA}[P](h') \text{ or } \varphi_A^{NDA}[P'](h') = \varphi_A^{NDA}[P](h').$$

Proof. We proceed by contradiction, i.e. we assume that $aP_{h'}a'$ for some $h' \in H$, where

$$\varphi_A^{NDA}[P'](h') = a' \text{ and } \varphi_A^{NDA}[P](h') = a \text{ for } h' \neq h.$$

So, there exists a step t of $NDA[P']$ at which h' is rejected from a. Hence, by Lemma 3.2, the apartment a is assigned to some household h''. We analyze the following cases.

Case 1. If $\tau(h'') = \tau(h') = i$. Since $\mu^{NDA}[P']$ is fair, we have that $(a, h'') \succ^i (a, h')$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

Case 1.1 $aP_{h''}a''$. Then (h'', i) has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$.

Case 1.2 $a''P_{h''}a$. By Lemma 3.2, there exists a household h''' such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h''', which generates an infinite succession of households $\{h^{(k)}\}$. This is not possible because H is finite.

Case 2. If $i = \tau(h') \neq \tau(h'') = i'$. Since $\mu^{NDA}[P']$ is fair, we have that $i'\pi_a i$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

Case 2.1 $aP_{h''}a''$. Then (h'', i') has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$.

Case 2.2 $a''P_{h''}a$. By Lemma 3.2, there exists a household h''' such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h''', which generates an infinite succession of households $\{h^{(k)}\}$. This is not possible because H is finite.

Therefore, no household h' is worse off under the assignment $\mu^{NDA}[P']$. □

We are ready to prove that truth-telling is a dominant strategy for all households. We assume, on the contrary, the existence of household h^* and a successful misrepresentation P'_{h^*} of P_{h^*}. That is to say

$$a' P_{h^*} a,$$

where $\varphi_A^{NDA}[P](h^*) = a$ and $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}] = a'$

By Lemma A.1, we consider that P_{h^*} is the preference list where a' is the most preferred apartment. Our objective is to show that P'_{h^*} is not a successful manipulation. To do that, we follow the proof of Roth (1982) about the strategy proofness of the DA. We say that household h **makes a match** at step t of the NDA algorithm, if h demands $\varphi_A^{NDA}(h)$ at step t. This proof analizes the possible steps where h^* makes a match.

First, if a household makes a match in the last step of the NDA algorithm, under the true preferences, then no manipulation is a successful misrepresentation of her true preference list.

Claim A.1. If h^* makes a match during the last step of the NDA, then there is no profitable deviation P'_{h*} of her true preference list P_{h^*}.

Proof. Let T be the last step of the $NDA[P]$ and consider a household h^* that makes a match at step T, say $a = \varphi_A^{NDA}(h^*)$. Since μ^{NDA} is non-wasteful, all apartments are matched and at $T-1$ either

1. a is unmatched, or

2. a is matched to a household h_1 who is unmatched at $\mu^{NDA}[P]$.

Case 1. Since apartment a was unmatched at $T-1$, all matched households prefer their match at μ^{NDA} to a. By Lemma A.2, this implies that none of them get a at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$. So, h^* does not strictly improves her match declaring P'_{h^*}.

Case 2. Let h_1 be the household matched with a at $T-1$, who is unmatched at T. For all matched $h_2 \neq h_1$, if any, who prefer a to $\varphi_A^{NDA}[P](h_2)$, we have that $(a, h_1) \succsim_{\tau(h_1)} (a, h_2)$ or $\tau(h_1)\pi_a\tau(h_2)$. Thus, if h^* strictly improves declaring P'_{h^*}, then h_1, or an unmatched household, gets a or an apartment preferred to a because the assignment is fair; in contradiction to Lemma A.2.

In both cases we conclude that $\varphi_A^{NDA}[P'_{h^*}, P_{h^*}](h^*) = \varphi_A[P](h^*)$. □

Now, we consider that h^* makes a match at some step t of the $NDA[P]$ procedure, with $1 \leq t < T$. We show that no household, matched after t, changes its final allocation when household h^* misrepresents her true preference list through P'_{h^*}.

Claim A.2. Consider a household h^* with preferences P_{h^*} and P'_{h^*} such that

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

If $\varphi_A^{NDA}[P](h^0) = \varnothing$ then $\varphi_A^{NDA}[P](h^0) = \varnothing$.

Proof. By contradiction, suppose that h^0 gets an apartment at P'. Since assignments $\mu^{NDA}[P]$ and $\mu^{NDA}[P']$ are non-wasteful, this means that some household h, previously matched at P, is unmatched at P', violating Lemma A.2. □

Claim A.3. Suppose h^* makes a match at t^*, $1 \leq t^* \leq T$ and P_{h^*} and P_{h^*} such that

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

Then $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = \varphi_A^{NDA}[P](h^*)$ for h^t who makes a match at $t^* < t \leq T$.

Proof. The proof is by induction.

Base of Induction. Starts in $t = T$. Same arguments as Claim A.1.

Hypothesis of Induction. Suppose the property is true until step $t+1$.

Induction Step. Let a^t be the march of h^t at $\varphi_A^{NDA}[P](h^t)$.

Case 1. a^t is unmatched at $t-1$. Since a^t is matched at $t-1$, all households matched before/at t strictly prefer their match to a^t, by Lemma A.2 they do not get a^t at P'. By induction hypothesis and Claim A.2, those who make a match after t get the same apartment or nothing.

Case 2. a^t is matched at $t-1$. Let h^{t-1} be the match of a^t at $t-1$; thus h^{t-1} has top priority among households who prefer a^t to their match and make a match before/at t. By Claim A.2, fairness of $\varphi_A^{NDA}[P']$ and Lemma A.2, h^{t-1} should get a^t at $\varphi_A^{NDA}[P']$. Since h^{t-1} makes a match, if any, after t, it is not the case. □

Claim A.4. If a household h^* makes a match at step t, with $1 \leq t < T$, then

$$\varphi_A^{NDA}[P'](h) = \varphi_A^{NDA}[P](h)$$

for all h who makes a match at s, $t \leq s < T$.

Proof. **Base of Induction**. The induction starts at $T-1$.

Case 1. $\varphi_A^{NDA}[P](h^*)$ was unmatched at $T-2$. Since the apartment was unmatched at $T-2$, all households who made a match at/before $T-1$ prefer their match to this apartment. By Claims

A.2 and A.3, households who make a match at T or stay unmatched have the same match at $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}]$ and $\varphi_A^{NDA}[P]$. By Lemma A.2, the result follows.

Case 2. $a^* := \varphi_A^{NDA}[P](h^*)$ was matched at $T-2$ with household h_1.

Let H_1^T be the set of households who prefer a^* to their match at $\varphi_A^{NDA}[P]$, and make a match at T or stay unmatched. We have seen that $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h) = \varphi_A^{NDA}[P](h)$ for all $h \in H_1^T$. Let H_1^{T-1} be the set of households who prefer a^* to their match at $\varphi_A^{NDA}[P]$ and make a match before/at $T-1$.

We note that for all $h' \in H_1^{T-1}$, either $\tau(h^*) = \tau(h')$, then $(a^*, h^*) \succ_{\tau(h^*)} (a^*, h')$, or $\tau(h^*)\pi_{a^*}(h')$. So, by fairness of $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$, none of the households in H_1^T gets a^*, else h^* would have justified envy over (a, h). If h^* strictly improves declaring P'_{h^*}, then h_1 gets a^* or an apartment preferred to a^* by fairness of $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$, which is not possible since $h_1 \in H_1^T$. Thus, we have that h_1 demands a^*, but h^* displaces her from a^*. Therefore, $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}] = a^*$.

Induction Hypothesis. For each household h^* that makes a match at step t, with $1 < t < T$ or later, we have that $\varphi_A^{NDA}[P'](h) = \varphi_A^{NDA}[P](h)$ for all households h who make a match at $t \leq s < T$.

Induction step. We have the following cases.

Case 1. $\varphi_A^{NDA}[P](h^*)$ was unmatched at $t-2$. Since the apartment was unmatched at $t-2$, all households who made a match at/before $t-1$ prefer their match to this apartment. By the induction hypothesis and Claim A.2, households who make a match at $t \leq s < T$ or stay unmatched have the same match at $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}]$ and $\varphi_A^{NDA}[P]$. By Lemma A.2, the result follows.

Case 2. $a^* := \varphi_A^{NDA}[P](h^*)$ was matched at $t-2$ with household h_1.

Let H_1^t be the set of households who prefer a^* to their match at $\varphi_A^{NDA}[P]$, and make a match at $t \leq s < T$ or stay unmatched. By induction hypothesis, we have that $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h) = \varphi_A^{NDA}[P](h)$ for all $h \in H_1^t$. Let H_1^{t-1} be the set of households who prefer a^* to their match at $\varphi_A^{NDA}[P]$ and make a match before/at $t-1$.

We note that for all $h' \in H_1^{t-1}$, either $\tau(h^*) = \tau(h')$, then $(a^*, h^*) \succ_{\tau(h^*)} (a^*, h')$, or $\tau(h^*)\pi_{a^*}(h')$. So, by fairness of $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$, none of the households in H_1^t gets a^*, else h^* would have justified envy over (a, h). If h^* strictly improves declaring P'_{h^*}, then h_1 gets a^* or an apartment preferred to a^* by fairness of $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$, which is not possible since $h_1 \in H_1^t$. Thus, we have that h_1 demands a^*, but h^* displaces her from a^*. Therefore, $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}] = a^*$.

In any step where household h^* makes a match, Claims A.1 and A.4 imply that $\varphi_A^{NDA}[P'](h^*) = \varphi_A^{NDA}[P](h^*)$. That is to say, there is no successful misrepresentation of P_{h^*}, and the NDA is strategy-proof.

□

Proof of Theorem 4.1 : Let x^* be the last iteration of the NDAI mechanism, i.e. there are no interrupter at the end of the NDA phase $x^*.1$. The last step of the NDA phase is denoted by T.

Distributional Constraints. Consider an institution i that does not fulfil its quota in the assignment μ^{NDAI}. Thus, $|\mu^{NDAI}(i)| < q_i$, i.e. there is at least one apartment a that remains unassigned. Since the over-demand condition holds in the market E, there exists an unassigned pair (a, h) that is acceptable for institution i where $aP_h\varnothing$. Considering the NDA algorithm during the Stage $x^*.1$ we have that (a, h) belongs to M_i^t, for some step t, or not.

Case 1. If $(a, h) \in M_i^t$ but $(a, h) \notin \mu^{NDAI}(i)$ this implies the existence of an institution j such that $j\pi_a i$, i.e. $a \in \theta(j)^{t'}$ for some step $t' \geq t$. However, we know that $a \notin \mu^{NDAI}(j)$. Hence, the institution j is a type 1 interrupter for a which contradicts the fact that there are no interrupters at x^*.

Case 2. If $(a, h) \notin M_i^t$, we have two possibilities. The first one, (a, h) is not acceptable for the institution i which contradicts the over-demand assumption. The second one, (a, h) does not belong to M_i^t because the institution i was an interrupter for a at some iteration $x' < x^*$. So, there exists an institution j and a household h_j such that $(a, h^j) \notin \mu^{NDAI}(j)$ during the Stage $x'.1$.

Case 2.1 The institution j is not an interrupter for apartment a. Then the construction of the NDAI assignment procedure implies that $a \in \theta^{NDAI}(j)$ in all iterations $x \geq x'$ of the NDAI algorithm. Therefore, $a \in \theta^{NDAI}(j)$ at Stage $x^*.1$ which contradicts the fact that $\mu^{NDAI}(a) = \varnothing$.

Case 2.2 The institution j is an interrupter for apartment a. So, there exists an institution j' and a household $h_{j'}$ such that $(a, h_{j'}) \notin \mu^{NDAI}(j')$ during the Stage $x''.1$, with $x'' < x'$. So, we apply the same reasoning on institution j' which generates an infinite succession of institutions $\{j^{(k)}\}$. This is not possible because I is finite.

In any case we get a contradiction. Therefore, we have that $|\mu^{NDAI}(i)| = q_i$ for all $i \in I$.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy. Let x^* be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup (a, h), q_i)$,

iii. $i\pi_a i'$.

By condition i), household h demands the apartment a at some step t. Moreover, the condition ii) ensures that the pair (a, h) is acceptable for the institution i, i.e. $(a, h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analize the following cases.

Case I. $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDA}(h') = a$, household h' demands the apartment a at some step t'. Even more, $(a, h') \in Ch_i(M_i^{t'}, q_i)$ because (a, h') belongs to the set $\mu^{NDA}(i)$. Given that $(a, h) \in M_i^t$ but $\varphi_A^{NDA}(h) \neq a$, the definition of the choice function ensures that

$$(a, h') \succ^i (a, h). \tag{6}$$

Moreover, since no apartment can be paired twice and the fact that priorities are responsive, the condition ii) implies that

$$(a, h) \succ^i (a, h'),$$

in contradiction with (**??**).

Case II. $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$. We know that $a \in \theta^{NDA}(i')$ and there are no interrupters during the iteration x^*, then there exists a step t' where

$$i'\pi_a j \text{ for all } j \in I_a^t, \text{ for all } t \geq t',$$

according to the Phase B.3. In particular

$$i'\pi_a i,$$

in contradiction with the condition iii).

In any case we get a contradiction, therefore there is no justified envy at the assignment μ^{NDA}. So, this assignment is fair.

Pareto Undominated for fair assignments. We proceed by contradiction, i.e. we assume the existence of a fair assignment $\mu = (\varphi^\mu, \theta^\mu)$ such that all households h weakly prefer $\varphi_A^\mu(h)$ to

$\varphi_A^{NDAI}(h)$. So, all households h did an offer to $\varphi_A^{\mu}(h)$ before to do it to $\varphi_A^{NDAI}(h)$ and have been rejected.

Let t be the first step where household h_1 is rejected from (a, i_1) where $a = \varphi_A^{\mu}(h_1)$ and $\tau(h_1) = i_1$. Then there exists (h_2, i_2) such that $\varphi^t(h_2) = (a, i_2)$.

Case 1. $i_1 = i_2$. Given that $\varphi^t(h_2) = (a, i_2)$, then $(a, h_2) \succ^{i_2} (a, h_1)$. Thus, it cannot be that $\varphi_A^{\mu}(h_2) P_{h_2} a$ because t is the first step where a household is rejected from an apartment, i.e. h_2 is never rejected from $\varphi_A^{\mu}(h_2)$. So, (h_2, i_1) has justified envy over the pair (h_1, i_1) at the apartment a in the assignment μ.

Case 2. $i_1 \neq i_2$. Given that $\mu^t(a) = (h_2, i_2)$, we have that $i_2 \pi_a i_1$. Moreover, h_1 is the first one to be dropped from its house at the NDAI. So $a P_{h_2} \varphi_A^{NDAI}(h_2)$. Since priorities are responsive, we conclude that (h_2, i_2) has justified envy over the pair (h_1, i_1) at the apartment a in the assignment μ.

Strategy Proofness. First, we prove the following proposition.

Claim A.5. Consider a market through institutions with distributional constraints where the over-demand condition holds. Consider a pair (a, h) such that $a P_h \varphi_A^{NDAI}[P](h)$ and $i = \tau(h)$. There is no misrepresentation P'_h of P_h such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = a$.

Proof. Let P'_h be a misrepresentation of P_h where apartment a is acceptable. We analyze the following cases.

Case 1. The institution i is an interrupter for apartment a at some iteration x. As a consequence, all pairs (a, h) are deleted from the preference $\succ^{i,x}$, i.e., pairs (a, h) are not acceptable at preference $\succ^{i,x'}$ for all $x' > x$. So, household h is not assigned to apartment a in stages $x'.1$, with $x' > x$. Therefore, P'_h is not a successful misrepresentation of P_h.

Case 2. The institution i is not an interrupter for apartment a. It is important to note that in a market that satisfies the over-demand condition, the assignment μ^{NDAI} satisfies the distributional constraints, so, it is non-wasteful. Hence, the conclusion of Lemma 3.2 still holds in an over demanded market. Then, we proceed as in Theorem 3.1. So, no misrepresentation of P_h is successful.

□

Therefore, the NDAI mechanism is strategy-proof.

B Markets where agents are attached to multiple institutions

B.1 The NDAI satisfies classical properties of the deferred acceptance algorithm

We first state the extension of Theorem 4.1 to the case where agents are attached to multiple institutions:

Theorem B.1. *Consider a matching market with institutions* $E = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$ *where each institution is over demanded and households can belong to many institutions.*

1. *The* μ^{NDAI} *is individually rational, non-wasteful, respects distributional constraints and there is no justified envy; namely, the assignment* μ^{NDAI} *is fair.*

2. *There is no fair assignment that Pareto dominates* μ^{NDAI}*.*

3. *The NDAI is strategy-proof for households.*

Proof of Theorem B.1:

Proof. **Distributional Constraints**. See the proof of Theorem 4.1.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy. Let x^* be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $i \in \tau(h)$ and $i' \in \tau(h')$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup (a, h), q_i)$,

iii. $i\pi_a i'$.

The proof is analogous to the proof of Theorem 4.3.

Pareto Undominated for fair assignments. We proceed by contradiction, i.e. we assume the existence of a fair assignment $\mu = (\varphi^\mu, \theta^\mu)$ such that all households h weakly prefer $\varphi_A^\mu(h)$ to

$\varphi_A^{NDAI}(h)$. So, all households h did an offer to $\varphi_A^{\mu}(h)$ before to do it to $\varphi_A^{NDAI}(h)$ and have been rejected.

Let t be the first step where household h_1 is rejected from (a, i_1) where $a = \varphi_A^{\mu}(h_1)$ and $i_1 \in \tau(h_1)$. Then there exists (h_2, i_2) such that $\varphi^t(h_2) = (a, i_2)$.

Case 1. $i_1 \in \tau(h_2)$. Given that $\varphi^t(h_2) = (a, i_2)$, then $(a, h_2) \succ^i (a, h_1)$. Thus, it cannot be that $\varphi_A^{\mu}(h_2) P_{h_2} a$ because t is the first step where a household is rejected from an apartment, i.e. h_2 is never rejected from $\varphi_A^{\mu}(h_2)$. So, (h_2, i_1) has justified envy over the pair (h_1, i_1) at the apartment a in the assignment μ.

Case 2. $i_1 \neq i_2$. Given that $\mu^t(a) = (h_2, i_2)$, we have that $i_2 \pi_a i_1$. Moreover, h_1 is the first one to be dropped from its house at the NDAI. So $a P_{h_2} \varphi_A^{NDAI}(h_2)$. Since priorities are responsive, we conclude that (h_2, i_2) has justified envy over the pair (h_1, i_1) at the apartment a in the assignment μ.

Strategy-proofnes. It follows from the following Claim.

Claim B.1. Consider a market through institutions with distributional constraints where the over-demand condition holds. Consider a pair (a, h) such that $a P_h \varphi_A^{NDAI}[P](h)$. There is no misrepresentation P'_h of P_h such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = a$.

Proof. Let P'_h be a misrepresentation of P_h where apartment a is acceptable. We analize the following cases.

Case 1. All institutions in $i \in \tau(h)$ are interrupters for the apartment a. As a consequence, all pairs (a, h) are deleted from the preference $\succ^{i,x_i}$, i.e., pairs (a, h) are not acceptable at preference $\succ^{i,x'}$ for all $x' > x_i$. So, household h is not assigned to apartment a in stages $x'.1$, with $x' > x_i$. Therefore, P'_h is not a successful misrepresentation of P_h.

Case 2. There exists an institution $i \in \tau(h)$ such that i is not an interrupter for apartment a. It is important to note that in a market that satisfies the over-demand condition, the assignment μ^{NDAI} satisfies the distributional constraints, so, it is non-wasteful. Hence, the conclusion of Lemma 3.2 still holds in an over demanded market. Then, we proceed as in Theorem 3.1 to prove that no misrepresentation of P_h is successful. □

Therefore, Claim B.1 implies that the NDAI algorithm is strategy-proof. □

B.2 Assignments satisfying distributional constraints do not exist

Next we extend Example 4.1 (showing that assignments satisfying distributional constraints do not necessarily exist to the case where agents belong to multiple institutions.

Example B.1. Let $I = \{1,2\}$, $\{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, where households type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_3, h_5\}$ and $\tau^{-1}(2) = \{h_1, h_4, h_5\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions priorities and households preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_3, h_4) \\ (a_3, h_2) & \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} \\ a_1 & a_1 & a_1 & a_1 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_2 & a_3 \end{pmatrix}.$$

As in the Example 4.1, no acceptable pair (a, h) include the apartment a_2. Consequently, there is no IR assignment that respects the distributional constraints. $\square$

In both examples, the number of households attached to each institution is greater than the institution quota. So, the size of the market is not enough to guarantee the existence of an assignment that respects the distributional constraints.

B.3 A sufficient condition for existence

To analyze the existence of assignments that respect the distributional conditions, we represent a market with institutions through a MTI hyper-graph.

Let $E = (I, H, \tau, D, Q, A, \delta, P, \succ, \pi)$ be a matching market through institutions. A **MTI hyper-graph** is an ordered pair $Y_{MTI} = (V(Y_{MTI}), HE[Y_{MTI}])$ where $V(Y_{MTI}) = I \cup A \cup H$ is the set of nodes and $HE[Y_{MTI}] \subset 2^{V[Y_{MTI}]}$ is the set of hyper-edges. A subset $\hat{e}$ of $V[Y_{MTI}]$ belongs to $HE(Y_{MTI})$ if and only if $\hat{e} \cap I = \{i\}$. That is to say, each hyper-edge of Y_{MTI} only has one institution. Since $I \cap (A \cup H) = \varnothing$ and $|\hat{e} \cap I| = 1$, we say that hyper-graphs Y_{MTI} are bipartite.

Example B.2. Consider the market $H = \{h_1, h_2, h_3, h_4, h_5\}$, $I = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4\}$. The type function is described by $\tau^{-1}(1) = \{h_1, h_2\}$, $\tau^{-1}(2) = \{h_3, h_4\}$ and $\tau^{-1}(3) = \{h_3, h_5\}$.

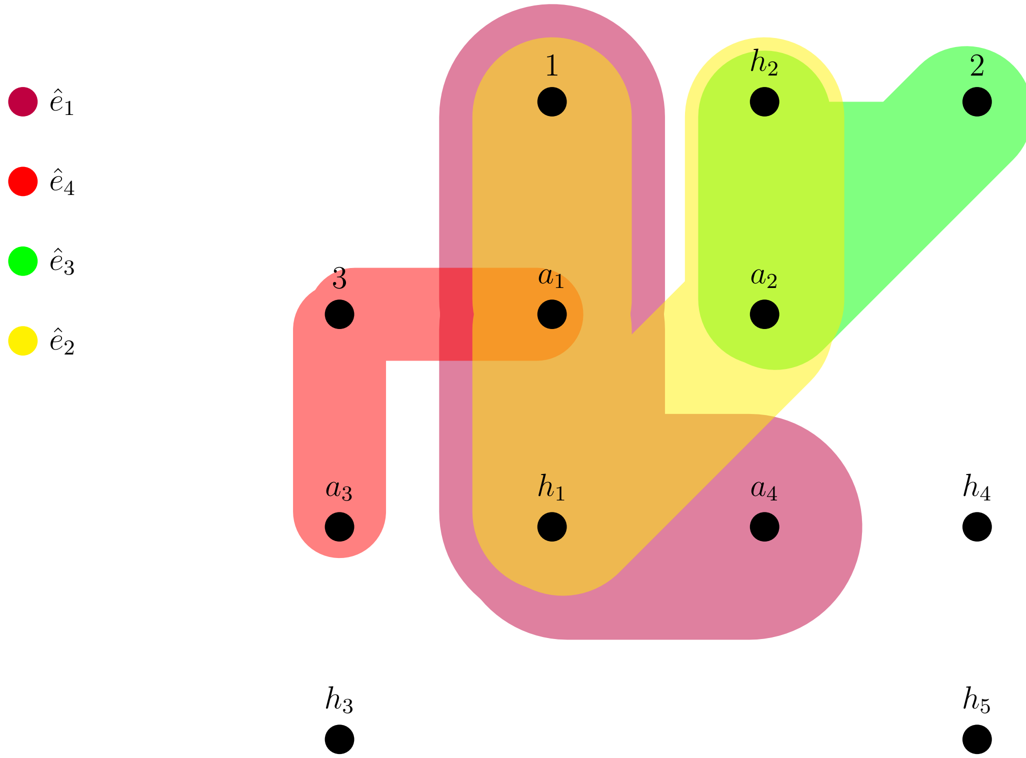

Figure 1: The institutions and pairs are represented by a node. The hyper-edges are illustrated as a box with its corresponding nodes.

We define the following hyper-edges.

$$\begin{aligned}\hat{e}_1 &= \{1, a_1, h_1, a_4\}\\ \hat{e}_2 &= \{1, a_1, h_1, a_2, h_2\}\\ \hat{e}_3 &= \{2, a_2, h_2\}\\ \hat{e}_4 &= \{3, a_1, a_3\}.\end{aligned}$$

The hyper-graph Y_{MTI} is illustrated in the Figure 1.

□

Considering a hyper-graph Y_{MTI}, we can restrict our attention to a subset of institutions and their relation with apartments and households. For a subset $C \subseteq I$, the set of hyper-edges that include some institution $c \in C$ is $HE_C = \{F \subseteq (A \cup H) : \{c\} \cup F \in HE[Y_{MTI}]$ for some $c \in C\}$. The **hyper-graph associated to** C and $(\mathbf{A} \cup \mathbf{H})$ is the hyper-graph $Y_C = (C \cup (A \cup H), HE_C)$.

A subset $\mathcal{M} \subseteq HE[Y_{MTI}]$ is a **matching** if no pair of hyper-edges in $\mathcal{M}$ have a node in common. A node $v \in V[Y_{MTI}]$ is **covered** by the matching $\mathcal{M}$ if v is an element of some hyper-edge of

$\mathcal{M}$. A matching is **perfect** if it covers all the nodes in the bipartite hyper-graph Y_{MTI}. In the Example 1, households h_3, h_4 and h_5 are not covered by a hyper-edge. So, a perfect matching does not exist. We denote by $\eta[Y_{MTI}]$ the maximum cardinality of a matching of Y_{MTI}.

A **transversal** is a subset $T \subseteq V[Y_{MTI}]$ with the property that $E \cap T \neq \varnothing$ for all $E \subseteq HE$. Let $\tau[Y_{MTI}]$ be the maximum cardinality of a transversal of Y_{MTI}. In the Example 1, it is easy to check that the set $\{a_2, h_1, a_1\}$ is a transversal.

It is important to note that a matching of a hyper-graph Y_{MTI} determines an assignment between households, apartments, and institutions. However, this kind of hyper-graphs allows hyper-edges with one institution, apartments and no households, see hyper-edge e_4 in Figure 1. That is to say, a matching of Y_{MTI} is not necessarily an IR assignment. So, we focus in a subfamily of hyper-graphs Y_{MTI}, the HAI hyper-graphs.

Let $M_i = \{(a,h) \in A \times H | (a,h) \succ^i \varnothing\}$ be the set of all acceptable pairs for the institution i. An **households-apartments-institutions** hyper-graph, or simple HAI hyper-graph, is an ordered pair $\Delta = (V[\Delta], E[\Delta])$ where the set of nodes is $V[\Delta] = I \cup (A \cup H)$, and $E[\Delta]$ denotes the set of hyper-edges $\hat{e}$. A hyper-edge $\hat{e} \in E[\Delta]$ if and only if $|\hat{e}| = q_i$, for some $i \in I$, and for all apartment a/household h in $\hat{e}$ there exists a unique households h/apartment a such that $(a,h) \in M_i$. In other words, a HAI hyper-graph is a hyper graph Y_{MTI} where each hyper-edge is a subset of $\{i\} \times 2^{M_i}$ with cardinality q_i, for some institution i.

Example B.3. Consider the market $H = \{h_1, h_2, h_3, h_4, h_5\}$, $I = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4\}$. The type function is described by $\tau^{-1}(1) = \{h_1, h_2\}$, $\tau^{-1}(2) = \{h_3, h_4\}$ and $\tau^{-1}(3) = \{h_3, h_5\}$; and the vector of quotas is $Q = (2, 1, 1)$. The type function is given by $\tau^{-1}(1) = \{h_1, h_2\}$, $\tau^{-2}(2) = \{h_3, h_4\}$ and $\tau^{-1} = \{h_5\}$. The institutions priorities are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 & \succ^3 \\ (a_1, h1) & (a_1, h_3) & (a_1, h_5) \\ (a_1, h2) & (a_2, h_3) & (a_2, h_5) \\ (a_2, h1) & (a_3, h_4) & (a_3, h_5) \\ (a_2, h2) & (a_1, h_4) & \\ (a_3, h1) & (a_2, h_4) & \end{pmatrix}.$$

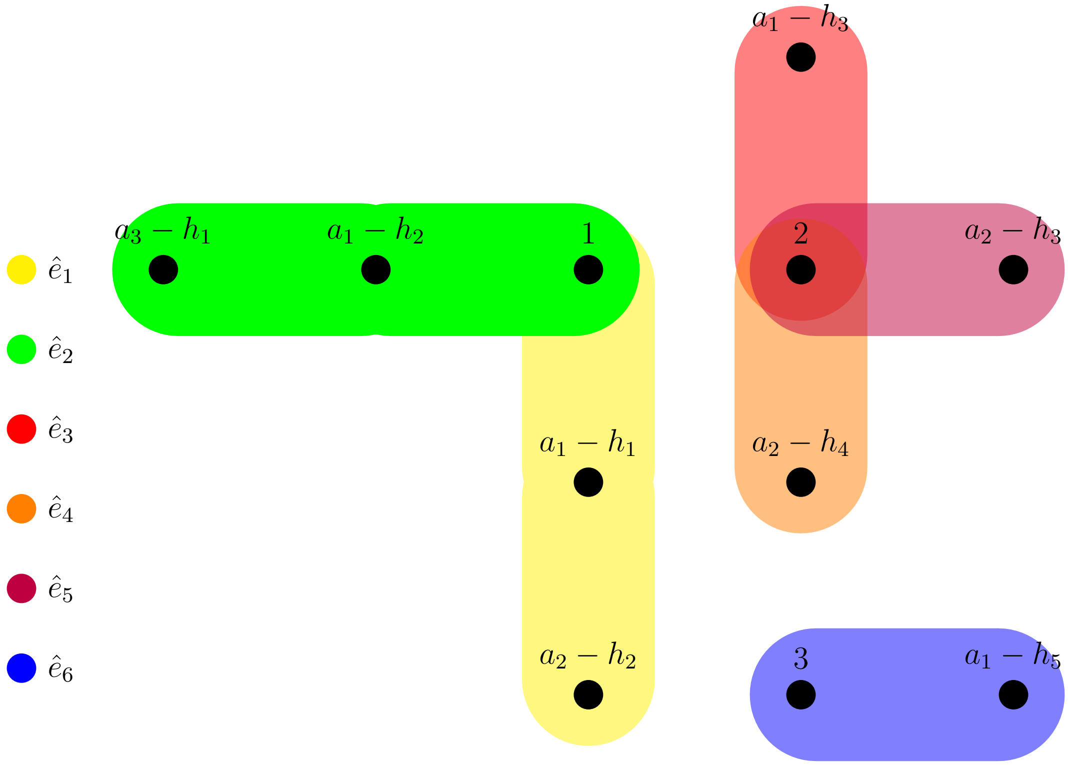

Figure 2: A HAI hyper-graph is a hyper-graph where each hyper-edge has one institution, q_i apartments and q_i households. Each apartment is linked to one and only one household.

We define the following hyper-edges.

$$\begin{aligned}
\hat{e}_1 &= \{1, (a_1, h_1), (a_2, h_2)\} \\
\hat{e}_2 &= \{1, (a_1, h_2), (a_3, h_1)\} \\
\hat{e}_3 &= \{2, (a_1, h_3)\} \\
\hat{e}_4 &= \{2, (a_2, h_4)\} \\
\hat{e}_5 &= \{2, (a_2, h_3)\} \\
\hat{e}_6 &= \{3, (a_1, h_5)\}.
\end{aligned}$$

Thus, each hyper-edge is composed by q_i acceptable pairs (a, h). That is to say, we have a HAI hypergraph, see Figure 2. $\square$

Finding an assignment that satisfies the distributional constraints is equivalent to find a matching that covers all institutions in the corresponding HAI hyper-graph.

For simple graphs, the Hall's theorem establishes a necessary and sufficient condition to guarantee the existence of a perfect matching in simple bipartite graphs.[5]. The generalization of the Hall's Theorem for hyper-graphs remains an open question. However, there exist sufficient, but not necessary, conditions that determine the existence of a perfect matching in a hyper-graph. One of them is related with the Haxell's condition.

Let $r \geq 2$. A bipartite hyper-graph $Y = (Y_1 \cup Y_2, E[Y])$, with $Y_1 \cap Y_2 = \varnothing$, satisfies the **Haxell's condition** if $|\hat{e} \cap Y_1| = 1$ and $|\hat{e} \cap Y_2| \leq r - 1$ for all $\hat{e} \in HE$.

Proposition B.1. *The HAI hyper-graphs satisfies the Haxell's condition.*

Proof. By definition, a hyper-edge $\hat{e} \in HE[\Delta]$ is a subset of $\{i\} \times M_i$ for some $i \in I$. Then, we have that $|\hat{e} \cap I| = 1$ and $|\hat{e} \cap (A \cup H)| \leq 2q^* + 1 - 1$ for all $\hat{e} \in HE$, where $q^* = \max_{i \in I}\{q_i\}$. □

Since HAI hyper-graphs satisfy the Haxell's condition, we apply the Haxell's Theorem to guarantee the existence of a matching that covers all institutions.

Theorem B.2. *If $\Delta = (I \cup A \cup H, HE[\Delta])$ is a HAI hyper-graph and $\tau(Y_C) > (4q^* - 3)(|C| - 1)$ for every $C \subseteq I$ and $q^* = \max\{q_i | i \in I\}$, then there exists an IR assignment that satisfies the distributional constraints.*

Proof. Given that HAI hyper-graph satisfy the Haxell's condition, we apply Theorem 3 of [17]. That is to say, there exists a matching that covers all the nodes in I. By construction of HAI hyper-graphs, each hyper-edge of a HAI hyper-graph is a subset of q_i acceptable pairs (a, h) assigned to the institution in the hyper-edge. Therefore, there exists at least one assignment that respects the distributional constraints. □

B.4 NDA does not output an assignment satisfying distributional constraints

The next Example extends example 4.2 showing that the NDA does not necessarily output an assignment satisfying the distributional constraints when agents are attached to multiple institutions.

[5]**Hall's Theorem**. Consider a bipartite hyper-graph Y such that $|\hat{e}| = 1$ for all $\hat{e} \in E[Y]$. A perfect matching exists if and only if $|S| \leq N_Y(S)$ for all $S \subseteq N[Y]$, where $N_Y(S)$ is the set of nodes that have an edge in common with some node in S.

I	$H^1_{a_1,i}$	$H^1_{a_2,i}$	$H^1_{a_3,i}$
1	h_1, h_2, h_3	$\varnothing$	$\varnothing$
2	h_2	$\varnothing$	$\varnothing$

Table 14: A. Elicited Demand Of Households step 1

Example B.4. (**No distributional constraints with multiple types**). Consider $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3\}$, and the vector of quotas is $Q = (q^1, q^2) = (2, 1)$. The type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_3\}$ and $\tau^{-1}(2) = \{h_2\}$. The profiles of institutions priorities, households preference and apartments priorities are

$$\begin{pmatrix} \succ^1 & \succ^2 \\ (a_1,h_1) & (a_1,h_2) \\ (a_1,h_2) & (a_2,h_2) \\ (a_1,h_3) & (a_3,h_2) \\ (a_2,h_1) & \\ (a_2,h_2) & \\ (a_2,h_3) & \\ (a_3,h_1) & \\ (a_3,h_2) & \\ (a_3,h_3) & \\ & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{pmatrix}, \qquad \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Running the NDA algorithm, part A of the step 1 is described in Table 14. We have that institution 1 demands apartment 1, and institutions 1 and 2 demand the apartment a_2 where $1\pi_{a_2}2$. The assignment induced by institutions priorities is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \varnothing \\ \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

At step 1, household h_3 is rejected from the apartment a_2. So, the NDA algorithm goes to step 2, and phase A of this step is summarized in Table 15. According to preference $\succ^1$, we have that $Ch^2_1 = \{(a_1, h_1), (a_2, h_2)\}$ and $Ch^2_2\{(a_2, h_2)\}$. Moreover $1, 2 \in I^2_{a_2}$ and $1\pi_{a_2}2$. Consequently,

$a_2 \in \theta^2(1)$, and the tentative assignment produced at the end of the step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \varnothing \\ \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$
1	h_1	h_2, h_3	$\varnothing$
2	$\varnothing$	h_2	$\varnothing$

Table 15: A. Elicited Demand Of Households step 2

At step 3, household h_3 asks for apartment a_3, her last acceptable apartment, so $a_1, a_2, a_3 \in A_1^3$. We know that

$$(a_1, h) \succ^1 (a_3, h) \text{ and } (a_2, h) \succ^1 (a_3, h) \text{ for all } h \in H.$$

Therefore institution 1 is assigned to $\{(a_1, h_1), (a_2, h_2)\}$ because $1\pi_{a_1}2$, $1\pi_{a_2}2$ and $q_1 = 2$. At the end of the step 3, household h_3 has been rejected from all her acceptable apartments, and other households are assigned to some apartment. The NDA algorithm stops and outputs the following assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \varnothing \\ 1 & 1 & \varnothing \end{pmatrix}.$$

The distributional constraints are not satisfied by the previous assignment because $|\mu^{NDA}(2)| = 0 < q_2$. However, in this market there exists an assignment that satisfies distributional constraints

$$\mu' = \begin{pmatrix} h_1 & h_3 & h_2 \\ a_1 & a_3 & a_2 \\ 1 & 1 & 2 \end{pmatrix}.$$

□

B.5 The NDA is not strategy-proof for institutions

The following example extends Example 3.2 to the case where agents are attached to multiple institutions.

Example B.5. Consider a market such that $I = \{1, 2\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$ and $A = \{a_1, a_2, a_3, a_4\}$. The vector of quotas is $Q = (3, 1)$, and the type function is given by $\tau^{-1}(1) = \{h_1, h_2, h_3, h_4\}$ and $\tau^{-1}(2) = \{h_1, h_3, h_5\}$. The institutions priorities, households preferences and apartments priorities are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_3, h_3) \\ (a_2, h_2) & (a_1, h_1) \\ (a_3, h_3) & (a_1, h_3) \\ (a_4, h_4) & (a_3, h_5) \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} \\ a_1 & a_2 & a_3 & a_4 & a_1 \\ a_2 & a_3 & a_4 & a_3 & a_2 \\ a_3 & a_4 & a_1 & a_1 & a_3 \\ a_4 & a_1 & a_2 & a_2 & a_1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} & \pi_{a_4} \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

Running the NDA algorithm, we get the following assignment

$$\mu^{NDA}[\succ] = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \\ a_1 & a_2 & a_3 & a_4 & \varnothing \\ 2 & 1 & 1 & 1 & \varnothing \end{pmatrix}.$$

Now, consider that institution 1 has the following preference

$$\succ^{1'} = \begin{pmatrix} (a_1, h_1) \\ (a_2, h_2) \\ (a_4, h_4) \\ (a_3, h_3) \end{pmatrix}.$$

Running the NDA with priorities $\succ' = (\succ^{1'}, \succ^2)$, we get that

$$\mu^{NDA}[\succ'] = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \\ a_1 & a_2 & a_3 & a_4 & \varnothing \\ 1 & 1 & 2 & 1 & \varnothing \end{pmatrix}.$$

If institution 1 prefers $\{(a_1, h_1), (a_2, h_2), (a_4, h_4)\}$ to $\{(a_2, h_2), (a_3, h_3), (a_4, h_4)\}$, because (a_1, h_1) is preferred to any other acceptable pair (a, h), then the institution 1 can improve its final allocation by misreporting its priorities. Note that such assumption does not contradict the fact that priorities $\succ^i$ are responsive.