A Centralized Matching Market with Early Matches*

Guillaume Haeringer† Vincent Iehlé‡

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Abstract

A common practice for centralized clearinghouses is to compute, once and for all, the final allocation of students at a given authorized date. Sometimes, however, the centralized system proceeds sequentially by computing allocations at preliminary dates. At the authorized final date, the market is already partially cleared and remain active only the agents who still do hope (correctly or not) for better schools. This centralized one-time matching mechanism with multiple stages can account for an additional source of heterogeneity among students, like scheduling constraints, which has been ignored so far in the literature of matching under preferences. We model these unconventional clearinghouses to assess whether they succeed in maintaining the classical properties of two-sided matching. Our results can be used to evaluate (part of) the French system for college admissions where students can finalize their matches almost two months ahead the final date, if they want to.

Keywords: two-sided matching, dynamic matching, early match, multistage matching mechanism, decision timing, stability, school choice problem, scheduling constraints, French college admission, preference updating.

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†Baruch College.
‡Université Paris-Dauphine, PSL Research University, LEDa & CEREMADE.

1. Introduction

1.1. *Purpose*

Consider a centralized clearinghouse in charge of the enrollment in schools or colleges. A common practice is to compute, once and for all, the final allocation of students at a given *authorized* date (on the basis of students' preferences and schools' priorities).[1] Sometimes, however, the centralized system jumps the gun on its own and proceeds sequentially. It starts *matching earlier* by computing allocations for students at preliminary dates. Students can accept the proposal at any of these early dates so that, at the authorized final date, the market is already partially cleared and remain active only the students who still do hope (correctly or not) for better schools.[2]

Matching early can be profitable for students with *scheduling constraints* (e.g., for housing or moving purposes) by increasing their lists of acceptable choices in the early stages.[3] On the other hand, the students remaining "on the application", are possibly less vulnerable to scheduling constraints. Since they face a less competitive market in subsequent stages, a well-behaved assignment procedure should likely make them better off. Thus, multistage systems account for an additional source of heterogeneity among students, beyond the preferences over schools.[4] But accommodating another aspect of the students' characteristics may harm their successful functioning.

We propose a model of centralized matching with gradual acceptance dates to assess whether these unconventional clearinghouses maintain the expected properties of a two-sided matching. We define a class of *multistage matching mechanisms*, where, at every stage, every remaining student accepts the proposed match or declines it and updates his preferences for the next stage (possibly to account for time-dependent scheduling constraints). The priorities of schools are fixed once and for all at the beginning of the assignment procedure. Following a classical school choice perspective we assume that the proposed match at every

[1]It can be any legally binding date, e.g., right after the student body is officially known in the case of a mandatory exam.

[2]In practice, a student accepting an early match is *conditionally* accepted in that given school.

[3]Another plausible explanation for the existence of early matches relies on the existence of endogenous preferences. Students learns about their own preferences across time by looking at the successive allocations. But in that case, our model cannot explain why students should accept an early match.

[4]Obviously, this feature is not sustainable in the classical one-stage specifications that take place either too early with respect to the authorized date or too lately with respect to the scheduling constraints.

stage is obtained via *a one-stage stable mechanism* that takes as input a "spot" school choice problem (e.g., the deferred acceptance mechanism of Gale and Shapley (1962), see also Abdulkadiroğlu and Sönmez (2003)) for its school choice version). The outcome of the market that we study is obtained at the end of last stage by putting together the pairs of student/school matched at every stage. We evaluate the properties of the multistage system on the basis of this assignment, combined with the dates of proposals' acceptance and the latest preferences submitted by the students.

1.2. Results

The class of static mechanisms in which a two-sided matching is computed on the basis of preferences and priorities is thoroughly documented in the literature. In practice, especially for school choice problems, the most widely used is probably the deferred acceptance mechanism of Gale and Shapley (*DA mechanism* hereafter). The outcome of the mechanism satisfies the essential property of stability. In addition, the mechanism is robust against the manipulation of preferences (for one side of the market). Here the stable matching is computed at every stage in sub-markets induced by the students who are still on the market (i.e. those who declined the proposals in the previous stages). The presence of blocking pairs is thus less stringent as time goes on and, in general, the final allocation obtained by putting together the matched pairs is not stable. This leads us to define a slightly weaker notion of stability called *stability over early matches* that accounts for the sequentiality of the matching.

A matching (for the market with all participants) is stable over early matches if, first, it is individually rational with respect to the latest submitted preferences of the students (at the stage they accept the proposal). Second, at the stage a seat is accepted by a student, there were no other preferred school with vacant seats (according to his latest submitted preferences). Third, the matching satisfies a no envy condition but only over early matches, that is, if a student has justified envy over a pair composed by a student and a school then it must be the case that the pair was finalized later than the student's acceptance date.

Our main results deal with the stability over early matches of the multistage matching mechanisms that use stable one-stage mechanisms at every stage. Stability is obtained for two specifications of the spot mechanisms: the DA mechanism with students proposing and the one with colleges proposing. An interesting aspect of the results relies in the interplay between the property of stability and the degree of freedom left to students as they update their preferences. We are led to define a class a *refitting rules* which describe the set of allowable preferences that can be submitted at every stage and which depend on the previous proposal and the previous submitted preferences. Depending on the chosen version of the DA mechanism the restrictions we posit to get stability over early matches share all the same spirit: no school less preferred to a proposal at some stage can be preferred to that proposal later on.

A key property drives the proofs of our stability results. Under our restrictions on the refitting rules, we show that every student obtains a weakly better proposal than the former one at every stage, as long as the proposed match remains acceptable in the new ranking. Such a property is crucial in practice since it allows the clearinghouse to make credible promises at early stages. Essentially, it is as if the students who decline proposals are tenants of their proposals until the next proposal, as long as the proposal remains acceptable.

The stability results are not affected by the possibility of *dismissals* by students previously matched to schools in the system. For instances, such students may fail the entrance examination or join other competing centralized procedures during the period of time the multistage system takes place. If the ineligibility is notified for instance, the concerned students are removed from the system and the seats are put back in the pool of available seats for the next stages. Under the same maintained restrictions on the refitting rules, the system we describe turns out to be robust in confront of such withdraws. The intuition is simply that our solution concept considers only that backward looking envy so that the dismissals do not jeopardize our notion of stability.

The restrictions we posit on updating cover the case where students are not allowed to update their preferences. They submit preferences at the beginning of the first stage of the system and then choose at every stage whether they accept or decline the proposal. In our framework the set-valued refitting rule is simply given by the *identity mapping*. The multistage matching mechanisms with the identity mapping are most likely to be found in real life situations and allow for a straightforward welfare comparison between the multistage and one-stage versions. The multistage matching mechanism, with (students) DA algorithm as spot mechanism, is weakly preferred to its one-stage version by every student.[5]

The basic trade-off between being matched early and getting possibly a better seat lately

[5] Our restrictions also include the case schedule-preserving updating that allow the students to account for time dependent constraints.

is typically a strategic issue. In addition, students are also involved in a preference revelation game as they submit their new preferences at every stage. The overall mechanism is thus difficult to play by the students. Actually the multistage mechanism leaves room for the profitable manipulation of the student's preferences even for the most 'non manipulable" version of the spot mechanism based on the DA mechanism with students proposing and absence of updating across stages (identity mapping).

We establish a positive result if one considers the notion of *subgame perfect equilibrium* in the induced extensive game with perfect information and simultaneous moves. In that context, a profile of strategies where every student always submits the true preferences and all students accept at the same stage is an equilibrium. However the result is tight since profiles of strategies with differentiated acceptance dates cannot be sustained as an equilibrium in general.

Finally, the class of multistage matching mechanisms is designed to include a very stylized version of the *French system for college admissions*. There, the high-school students have three opportunities to finalize a match: either five or two weeks before or also one week after the results of the national examination.[6] In the meantime they also update their preferences across the stages within the centralized system. In addition, the one-stage matching mechanism used along the three stages is a stable one, namely the deferred acceptance mechanism with colleges proposing. We are thus in position to evaluate concretely multistage matching in a large scale application with no analogue elsewhere.[7]

We show that the French system, which allows for dismissals, always produces an allocation of students that is stable over early matches. In addition, the refitting rule offered to students guarantees that any kind of scheduling constraints can be integrated at every stage of the mechanism. Thus one of the conjectured goals of the multistage matching mechanisms is achieved. However, our theoretical analysis highlights the existence of simple profitable strategies based on the timing of decisions. It is likely that it makes the multistage system difficult to apprehend for the students contrary to more conventional one-stage mechanisms.[8]

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[6]For the student body we consider the exam is not very compelling and is not used for the rankings of the students, but still, it is a necessary requirement to enter the colleges (actually the preparatory classes to enter colleges...). Hence, the publication date of the exam's results stems for the so-called authorized date.

[7]In this article we document only one of the (numerous) specificities of the French system, the one regarding the admission to selective colleges. For non selective colleges, the admission process is based on lotteries in case of lack of vacancies. This latter issue is excluded from our analysis.

[8]This latter effect is difficult to measure from a theoretical viewpoint. It could be better assessed by

In our model, there is no sequential arrival of students. Even though, the matching mechanism is dynamic and not equivalent to any static mechanism with multiple rounds. The spot allocation computed at a given stage results from the acceptance decisions and the submitted preferences previous stages. Thus the dynamic dependence across stages is two-fold.

To our knowledge our model of one-time matching mechanism with multiple stages has not been studied by the recent body of literature dealing with dynamic matching. A common feature of two-sided dynamic matching models is indeed that agents are long-lived (possibly with overlapping generations) and may "consume" different matches at different periods of time. They can be either free to rematch or reassigned by a centralized clearing house at every stage (Kennes *et al.* (2014, 2015) and Pereyra (2013) among others). In contrast, we consider one-time matchings with multistage centralized mechanisms since the students are assigned at most once to a school but possibly at different periods. Baccara *et al.* (2015) and Doval (2016) consider also a model with irreversible one-time matches of agents arriving sequentially. But in essence their models are decentralized, and the focus is put on the trade-off between the waiting cost and higher quality match in the future.

Leaving aside many aspects, the model of Doval (2016) is probably the most related to ours if one looks at the timing of decision and the definition of a matching plan. She also proposes fairly general notion of blocking, based on the evaluation of future matching plans. In this regard, stability with early matches is less compelling and guarantees *only* a consistency requirement with respect to the past matches. Our solution concept actually reflects a practical limitation, inherent to centralized dynamic matching: agents can submit only spot preferences and not plain preferences over intertemporal matching plans.

One-time matchings with early matches recalls another stream of the matching literature dealing with the *unraveling phenomenon*. That is, the possibility to contract bilaterally before the job openings. Such strategic behavior has been identified first by Roth (1984) and documented by Roth (1991) and Kagel and Roth (2000) for specific labor markets. Halaburda (2010) and Echenique and Pereyra (2016) consider general matching markets to address this issue. In our framework, any early match is actually arranged by the centralized mechanism itself and does not involve any bilateral contract out of the system.[9] Nevertheless,

looking at the data.

[9] Avery and Levin (2010) consider a *decentralized* college admission with early stages of admissions that

the existence of multiple stages provides new and relatively accessible opportunities for manipulations that do not exist in static matching problems. This feature can be found in other contributions dealing with the DA mechanism in dynamic setting, such as Kennes *et al.* (2015) and Pereyra (2013). Even though our model is not directly comparable to theirs, our results and (counter)-examples appeal to similar arguments.[10]

We have invoked the existence of scheduling constraints to motivate the design of multistage matching mechanisms. Considering *endogenous preferences* is another promising route. Antler (2015) analyzes the structure of equilibria in presence of endogenous preferences in the context of a standard one-to-one matching model. Our setting where agents can update their preferences according to the past allocations and actions sounds adequate to make more explicit the formation of such preferences (or preferences resulting from early signals as in Avery and Levin (2010)).

Large centralized clearinghouses are present at many stages of the French educational system. Most of them are designed by the Ministry of (Higher) Education. They assign students to secondary schools, high schools, colleges/universities, graduate schools (business or engineering schools) and organize the recruitment (or transfer) of teachers and scholars. Admissions to high schools in most of the French regions are organized through the DA mechanism with schools proposing (Hiller and Tercieux, 2014). Assistant professors are assigned to departments through the DA mechanism with candidates proposing (Haeringer and Iehlé, 2010). French teachers are assigned according to a modified DA algorithm that accounts for individual initial endowments (Combe *et al.*, 2015). The French college admission system is the one operating at the largest scale (around 500,000 students per year) and the most complicated (or advanced?) among those designed by the Ministry of Education. To the best of our knowledge, it admits no analogue in practice. It can be partially described by a multistage matching mechanism.

The analysis is organized as follows. In Section 2 we recall the basic model of school choice we work with. In Section 3, we define the class of multistage matching mechanisms, based on school choice problems. Section 4 contains the main stability results for the deferred acceptance versions of the multistage matching mechanisms. The rest of the section is

are used by students to signal their interest to colleges.

[10]It also comes close to some of Kesten (2010)'s arguments showing the absence of strategyproofness of the efficient adjusted DA mechanism that is based on the removal of specific students during the rounds of the static DA mechanism.

devoted to some variations and generalizations of the model. We address in Section 5 the strategic issues (from the viewpoint of the students) that follow from the use multistage mechanisms. In Section 6 we apply our results to the French system for college admissions. The proofs of the main results are relegated in Appendix.

2. Preliminaries: the school choice problem

A **school choice problem** is given by a 5-tuple $(I, S, (P_s, q_s)_{s\in S}, (P_i)_{i\in I})$, where I is a set of students, S a set of schools, P_s a priority ordering for school s, $q_s \geq 0$ a capacity constraint for school s, and P_i a preference ordering for student i. Each school $s \in S$ is endowed with a fixed **capacity** of seats q_s and a **strict priority** ordering over the students, i.e., P_s is a linear ordering over I. We say that student i has **higher priority** (or is **ranked higher)** than student j at school s if iP_sj. Each student $i \in I$ has a **strict preference** relation P_i over the schools and the option of remaining unassigned, i.e., P_i is a linear ordering over $S \cup \{i\}$, where i denotes his outside option (e.g., going to a private school). The notation sP_is' means that student i prefers to go to school s than school s'. A school s is **acceptable** for a student i under the preferences P_i if i prefers to be matched to s than being matched to himself, i.e., sP_ii. We sometimes use the following notation $P_i = [s_1, s_2, \dots, s_j, i, \dots]$ denoting that student i's first choice is s_1, his second choice s_2 ($s_2P_is_1$) and so on, until at some point he prefers to remain unmatched. For every student i, let $\mathcal{P}_i$ be the set of all students' preferences over $S \cup \{i\}$. Given a preference relation P_i we denote by R_i the weak relation associated to it, i.e., $vR_iv' \;\Leftrightarrow\; vP_iv'$ or $v = v'$. For a set $J \subseteq I$, we denote by P_J the profile $(P_i)_{i\in J}$.

The problem studied here is that of matching students to schools in the limit of their capacities. A **matching** for a school choice problem $(I, S, (P_s, q_s)_{s\in S}, (P_i)_{i\in I})$ is a mapping $\mu : I \cup S \to 2^I \cup S$ such that, for each $i \in I$ and each $s \in S$,

- $\mu(i) \in S \cup \{i\}$,
- $\mu(s) \in 2^I$,
- $\mu(i) = s$ if, and only if, $i \in \mu(s)$, and
- $|\mu(s)| \leq q_s$.

For $v \in S \cup I$, we call $\mu(v)$ agent v's assignment. For $i \in I$, if $\mu(i) = s \in S$ then student i is **matched** to school s under μ. If $\mu(i) = i$ then student i is said to be **unmatched** under μ.

A matching is stable if, for each student, all the schools he prefers to his assignment have exhausted their capacities with students that have higher priority, and he is matched to an acceptable school. Formally, a matching μ is **stable** for a school choice problem Γ if μ is a matching for Γ and

(a) it is **individually rational**, *i.e.*, for all $i \in I$, $\mu(i) R_i\, i$,

(b) it is **non wasteful**, *i.e.*, for all $i \in I$ and all $s \in S$, $sP_i\mu(i)$ implies $|\mu(s)| = q_s$,

(c) there is no **justified envy**, *i.e.*, for all $i, j \in I$ with $\mu(j) = s \in S$, $sP_i\mu(i)$ implies jP_si.

Given a school choice problem Γ, we denote the set of stable matchings by $\Sigma(\Gamma)$. For a matching μ, the pair (i, s) is a **blocking pair** (or that i and s **block** the matching μ) if either i prefers s to his assignment and $|\mu(s)| < q_s$, or student i has justified-envy against a student j and a school s (i.e., $\mu(j) = s$, iP_sj, and $sP_i\mu(i)$). A **one-stage matching mechanism** maps school choice problems to matchings for those problems. Given a school choice problem Γ, we denote the student (resp. school) optimal stable matching by $SOSM(\Gamma)$ (resp. $COSM(\Gamma)$).[11] Consistently with our notations, the shorthands SOSM and COSM denote the associated mechanisms.

Students' preferences over schools can be straightforwardly extended to preferences over matchings. We say that student i prefers the matching μ to the matching μ' if he prefers his assignment under μ to his assignment under μ'. Formally, $\mu P_i \mu'$ if, and only if, $\mu(i) P_i \mu'(i)$, and $\mu R_i \mu'$ if, and only if, $\neg(\mu' P_i \mu)$. In the remainder, we consider the optimality from the viewpoint of students. A matching μ' Pareto dominates a matching μ for the students if all students prefer μ' to μ and there is at least one student that strictly prefers μ' to μ. Formally, μ' **Pareto dominates** μ for the students if $\mu' R_i \mu$ for all $i \in I$ (resp. $s \in S$), and $\mu' P_{i'} \mu$ for some $i' \in I$ (resp. $\mu' P_{s'} \mu$ for some $s' \in S$).

3. The framework of multistage matching

Let the set of schools S and the profile of schools' priorities $(P_s)_{s \in S}$ be fixed once and for all. For any subset of students J, profile of capacities $q = (q_s)_{s \in S}$ and profile of students'

[11] Gale and Shapley (1962) and Abdulkadiroğlu and Sönmez (2003).

preferences $(\widetilde{P}_i)_{i\in J}$, the school choice problem $\big(J, S, (\widetilde{P}_i)_{i\in J}, (P_s, q_s)_{s\in S}\big)$ is summarized by $(\widetilde{P}, J, q)$.

In multistage matching mechanisms, full matchings for school choice problems are computed at each stage. Students can either accept definitively the proposal at a given stage, or decline the proposal and possibly update their preferences. Remaining students (with their new preferences) and seats determine then a new school choice problem in the next step. In the benchmark version of the multistage mechanism, we assume that the number of stages is finite.

3.1. *The general multistage matching mechanism*

We start with the simplest specification of the mechanism. It is enriched progressively in the next sub-sections.

Let θ be any one-stage matching mechanism for school choice problems.

The initial input is a school choice problem $\Gamma = (P^1, I^1, q^1)$ (P_i^1 is submitted by the student i). Let ν be the null matching for Γ.[12]

- Step t, $1 \leq t < T$. Set $\mu^t := \theta(P^t, I^t, q^t)$.

  For every $i \in I^t$:

  Accept: If student i accepts $\mu^t(i)$ then $\nu(i) := \mu^t(i)$; $q^{t+1}_{\nu(i)} := q^t_{\nu(i)} - 1$ if $\nu(i) \in S$; $t_i := t$.

  Decline: If student i declines $\mu^t(i)$ then P_i^{t+1} is submitted and $I^{t+1} := I^{t+1} \cup \{i\}$.

- Step T. Set $\mu^T := \theta(P^T, I^T, q^T)$.

  For every $i \in I^T$, $\nu(i) := \mu^T(i)$ and $t_i := T$.[13]

Clearly, the mapping ν obtained at the end of stage T defines a (feasible) matching for the initial problem with students I^1 and the vector of capacities q^1. At each stage t a matching is computed, based on the submitted preferences P^t, the set of remaining students I^t, the available seats q^t.[14] Upon rejection student i submits P_i^{t+1}, which is to belong to a list of

[12] That is, $\nu(i) = i$ for every $i \in I^1$.

[13] Students cannot postpone their decisions in the last stage T, that is, we do as if they accept the proposed seats.

[14] If $q_s^t = 0$ the school s is not removed from the market. It is innocuous for the concepts we consider hereafter. We adopt that convention to model easily the possibility of dismissals, see Section 3.2.

acceptable preferences (see Section 2.3). The definitive match of i is given by $\nu(i)$ while t_i is used to keep track of the date at which i is assigned to a school (i can be matched to himself).

The submitted preferences of student i consist of a list $(P_i^1, P_i^2, \ldots, P_i^{t_i})$, with $1 \leq t_i \leq T$. We adopt hereafter the following notation $\widehat{P}_i$ to describe the **path of submitted preferences**, that is,

$$\widehat{P}_i = (P_i^t)_{1 \leq t \leq t_i} \text{ and } \widehat{P}_J = (\widehat{P}_i)_{i \in J}$$

All the relevant information to produce the final matching ν is summarized by the initial problem Γ and the path of submitted preferences $\widehat{P}_I$ since the preferences' profiles also encapsulate the acceptance/rejection decisions via the dates t_is.

A **multistage matching mechanism** follows the above T-stage procedure and maps school choice problems and paths of submitted preferences to matchings for the initial school choice problems. It is denoted by $\mathcal{M}^\theta$, where θ is the one-stage mechanism used at each stage to define μ^t. For a given Γ and $\widehat{P}_I$, the **outcome** of the multistage mechanism $\mathcal{M}^\theta$ is $\nu^\theta(\Gamma, \widehat{P}_I)$. When there is no ambiguity the outcome is also simply denoted by ν.

3.2. Stability over early matches

By construction the output of the multistage matching mechanism is a matching for the initial school choice problem. Thus the notion of stable mechanism can be defined as well in such an environment. But even in the most favorable case where stable matchings are chosen at each stage and preferences are maintained identical across stages, there is no hope to get stability of the multistage mechanism in general. The reason is well known in one stage environments where the concatenation of stable matchings in disjoint submarkets does not form a stable matching for the whole market. In our context the submarkets are those defined at each stage by the remaining students (and seats).

In what follows we consider instead a weaker requirement of stability. Nevertheless, the definition is natural in the sense that it fully accounts for the sequentiality of the multistage matching mechanism.

Definition 1 Let θ be a one-stage mechanism. The multistage matching mechanism $\mathcal{M}^\theta$ is *stable over early matches* if, for any $\Gamma = (P^1, I^1, q^1)$ and $\widehat{P}_{I^1}$, the outcome ν satisfies the three conditions hold for every student $i \in I^1$

- individual rationality: $\nu(i) R_i^{t_i} i$
- provisional non wastefulness: if $s P_i^{t_i} \nu(i)$ then $|\{j \,:\, t_j < t_i, \nu(j) = s\}| + |\mu^{t_i}(s)| = q_s$
- no justified envy against early pairs: for all $j \in I$ with $\nu(j) = s \in S$ and $t_j \leq t_i$, $s P_i^{t_i} \nu(i)$ implies $j P_s i$.

The first condition is individual rationality with respect to the latest submitted preferences of the students. The second condition is a weaker requirement than non wastefulness in the problem Γ. It simply says that whenever i prefers s to his assignment then it must be necessarily the case that s has no vacant seats at the time the student i was definitely assigned (i.e., at stage t_i). The third condition is also a weaker requirement than the no justified envy condition. We allow envy from the student i only at the time is definitely assigned, so that blocking is possible over past or current assignments but not with respect to a subsequent pair assigned later on.

A simple sequential interpretation of the sequential solution concept is that claims are formulated by students only at the stage they are assigned to their schools, so that the assignments in the next stages cannot be part of their claims. In a distinct interpretation claims are formulated *a posteriori*, at the end of stage T. In that case, the notion of stability with early matches says that desisting from some claims is the price to pay to get a seat at an early stage. More precisely, students who have been assigned at date t cannot claim any seat assigned or left vacant at date $t' > t$.

3.3. Refitting rules

An important feature of the multistage matching mechanism is that, in general, stability with early matches cannot be guaranteed even if a stable matching μ^t is chosen at each stage.[15] This is due to the potential degree of freedom left to students when they update their preferences across stages. This effect can be mitigated by adding restrictions on the way students can resubmit their preferences are needed. We model those restrictions as follows.

Definition 2 A refitting rule is a set-valued mapping

$$F \,:\, (P, p) \in \mathcal{P}_o \times (S \cup \{o\}) \;\rightrightarrows F(P, p) \subset \mathcal{P}_o$$

[15] See examples in the next subsection.

where $\mathcal{P}_o$ is the set of all preferences' lists over $S \cup \{o\}$, with o being the outside option. Let $\mathcal{F}$ be the class of all such mappings.

The definition generates, with an abuse of notations, a uniform rule for the students. For a student i, $F(P_i, p)$ is used to describe the set of acceptable lists of preferences if the list P_i has been submitted at the previous stage and school p (or himself) has been proposed. Hence, if two students i, j submit the same preferences over schools, say $P_i = [s_1, s_2, i, \ldots]$ and $P_j = [s_1, s_2, j, \ldots]$, and obtain the same proposal p (or i and j), then they obtain the same acceptable lists of preferences $F_i(P_i, p) = F_j(P_j, p)$, up to the fact that o stands for either i or j. Note that the class of rules under consideration is also stage independent.

In the multistage matching mechanism we assume that every student who declines a match at some stage resubmits preferences that agree with the rule F. That is, at every stage $t < T$ in $\mathcal{M}^\theta$:

$$P_i^{t+1} \in F(P_i^t, \mu^t(i)), \;\; \forall i \in I^t \text{ declining } \mu^t(i) \tag{$\star$}$$

It worths noting that the possibility to resubmit preference has a bite only if the refitting is a set-valued mapping, that is, when the student can choose among at least two rankings. If the rule is single-valued then $(\Gamma, \widehat{\succ}_I)$ can simply described by the initial problem Γ and the dates of acceptance $(\bar{t}_i)_i$ since the spot preferences can be deduced without ambiguity at every step. It follows that the final assignment ν can be simply computed by using that minimal information. [16]

The next definition provides a first restriction on refitting rules.

Definition 3 The refitting rule satisfies assumption A (A-refitting) if for every ranking $P_i \in \mathcal{P}_i$ and proposal p such that pR_ii, it holds that

$$pP_is \Rightarrow iP_i's \;\; \forall P_i' \in F(P_i, p)$$

Remark 1 The simplest A-refitting rule is the (single-valued) **truncation mapping** below the proposal: for any $P_i := [s_1, s_2, \ldots, s_k, i, \ldots]$ and pR_ii

$$F(P_i, p) = \{[s_1, s_2, \ldots s_j, i, \ldots] \; : \; s_j = p\}$$

For such a single-valued refitting rule, no degree of freedom is left to students in the choice of their spot preferences.

[16]See also Subsection 4.1 for further analysis in this line.

The next condition is weaker than Assumption A. Namely, there is no new acceptable students above the spot proposal in the new submitted preference.

Definition 4 The refitting rule satisfies assumption B (B-refitting) if for every ranking $P_i \in \mathcal{P}_i$ and proposal p such that $pR_i i$, it holds that

$$pP_i s \Rightarrow p\widetilde{P}_i s \text{ or } i\widetilde{P}_i\, p, s \;\; \forall \widetilde{P}_i \in F(P_i, p)$$

Remark 2 The simplest refitting rule satisfying B is the **identity mapping**: for any $P_i := [s_1, s_2, \ldots, s_k, i, \ldots]$ and $pR_i i$

$$F(P_i, p) = \{P_i\}$$

Then, no degree of freedom is left to students as it is the case for the truncation mapping.

Remark 3 A-refitting implies B-refitting, but the converse is not true (consider for the above identity mapping that satisfies B- but not A-refitting).

4. Stability of DA multistage matching mechanisms

We complete now our description of multistage matching mechanisms by considering the two most commonly used selections of the set of stable matchings, namely the student optimal stable matching (SOSM) and the college optimal stable matching (COSM). Matching mechanisms where the spot mechanisms θ is either COSM or SOSM are called *DA multistage matching mechanisms* in the remainder. For this class, we provide positive results for stability over early matches and we relate the property to restrictions to be made on the refitting rule. We also consider hereafter the properties of the DA multistage matching mechanisms when the refitting rules are strongly restrictive (identity and truncation mappings). Finally, we show how the results extend to the case with dismissals.

4.1. Main results for stability

We start with COSM.

Theorem 1 *The multistage matching mechanism $\mathcal{M}^{COSM}$ is stable over early matches under A-refitting.*

$P^1_{i_1} = P^2_{i_1}$	$P^1_{i_2} = P^2_{i^2}$	$P^1_{i_3}$	$\ldots$	P_{s_1}	P_{s_2}	P_{s_3}	$\ldots$
$\vdots$	$\vdots$	$\vdots$		i_2	i_1	i_1	
s_1	s_2	s_3		i_1	i_2	i_3	
s_3	s_1			$\vdots$	$\vdots$	$\vdots$	
s_2							

Table 1: An instance of input for $\mathcal{M}$

Remark 4 Without assumption A on the refitting rule, stability with early matches does not hold in general. Consider for instance the case where a student i_1 does not accept the proposed school $\mu^t(i_1) = s_1$ such that $\ldots s_1 P^t_i s_3 P^t_i s_2 P^t_i i$, at some step t, while a student i_3 with lower priority than i_1 in s_3 is assigned s_3. It can be the case that student i_1 ends up with s_2 under COSM in a subsequent stage. But this might violate non justified envy as one checks stability with early matches. This configuration is impossible under A-refitting since s_3 is never acceptable for student i_1 after step t so that i_1 cannot have a justified envy over the pair (i_3, s_3). This is the intuition behind the proof of Theorem 1.

A more tractable counter-example shows that stability does not hold under a minimal violation of assumption A.

Example 1 The school choice problem is described in Table 1. The capacity of each school is equal to one. There are two stages: $T = 2$. At stage 1, μ^1 is such that $\mu^1(i_1) = s_1$, $\mu^1(i_2) = s_2$ and $\mu^1(i_3) = s_3$. The respective assignments of i_1, i_2, i_3 are the only ones compatible with stability given the preferences and priorities of the agents (assuming that schools with higher priority in i_1 and i_2's preferences are not feasible at any stable matching); hence it is the COSM. Suppose that i_1 and i_2 decline their respective proposals while i_3 accepts s_3. The remaining agents maintain their preferences identical in the next stage (hence assumption A is not satisfied in a minimal sense), and they receive μ^2 such that $\mu^2(i_1) = s_2$ and $\mu^2(i_2) = s_1$. Since the seat at s_3 is already assigned to i_3, $q^2_{s_3} = 0$. Then the matching μ^2 is stable and candidate for COSM at stage 2 according to the new preferences of the remaining two agents P^2_1, P^2_2. But stability over early matches is not satisfied since i_1, with preference P^2_1 has justified envy against (i_3, s_3).

In the previous example instability relies on the fact that $\mathcal{M}^{COSM}$ picks the least favorable stable matching at stage 2. In the next sections, we control better this issue by considering the SOSM selection. But it still dictates restrictions on the updating rule.

Note that the proof of Theorem 1 makes use of the stability property of COSM but not of its optimal property. In other words, the previous result holds for any stable mechanism θ. It is stated in the next proposition.

Proposition 1 *The multistage matching mechanism $\mathcal{M}^\theta$ is stable over early matches for any one-stage stable mechanism θ if F satisfies A-refitting.*

Consider now the case where the student optimal stable matching is chosen at each step of the mechanism. The next theorem exhibits a new set of restrictions for the refitting rule to make $\mathcal{M}^{SOSM}$ stable over early matches. The result establishes that stability over early matches is guaranteed under a slightly weaker requirement than A-refitting. What is more, those restrictions are also necessary for the stability over the early matches. Hence, we can fully characterize the degree of freedom that the central planner can leave to the students when they adjust their preferences.

Theorem 2 *Let $\mathcal{M}^{SOSM}$ be the multistage matching mechanism using SOSM as the one-stage mechanism and F as the refitting rule. The following statements are equivalent:*

1. *F satisfies B-refitting.*
2. *$\mathcal{M}^{SOSM}$ is stable over early matches.*

Recall that Theorem 1 with requires a different condition on the refitting rule. The intuition is that choosing COSM is the least favorable specification to get stability with early matches because the mechanism features two contradicting forces: COSM favors the schools while the readjustment of preferences gives room to students. Reconsidering Example 1 might help to get the idea. The B-refitting is satisfied since preferences are unchanged across dates (identity mapping). According to Theorem 2 the outcome $\nu^{SOSM}(\Gamma^1, \widehat{\succ}_I)$ is thus stable over early matches. Indeed, at date $t = 1$, the matching μ^1, where $\mu^1(i_1) = s_1$, $\mu^1(i_2) = s_2$ and $\mu^1(i_3) = s_3$, is the only compatible matching with SOSM. If i_1 and i_2 decline, SOSM outputs necessarily $\mu^2(i_1) = s_1$ and $\mu^2(i_2) = s_2$ at date 2, which is clearly stable over early matches.

Remark 5 The reader has also noticed that Theorem 1 establishes "only" a sufficient condition for stability over early matches. Whether the A-refitting is also necessary for stability in $\mathcal{M}^{COSM}$ is an open question. The necessary condition is likely to be more difficult to check than in the case of SOSM as it involves more restrictions

We obtain an additional property satisfied by the multistage matching mechanisms. Proposals made by the market maker at each stage can be viewed as promises with respect to future proposals. Indeed, every student declining a proposal at a given stage will be proposed a new assignment that is better or equal than the former one with respect to his new spot preferences. It is of importance in practice since the market designer can make a commitment about the future proposed seats that are "better or equal".[17] The proof of the next proposition is omitted since it is just a restatement of Lemmas 2 and 3 used respectively in the proofs of Theorems 1 and 2.

Proposition 2 *Under A-refitting (resp. B-refitting), at any step $\bar{t}$ of $\mathcal{M}^{COSM}$ (resp. $\mathcal{M}^{SOSM}$), it holds that for every student $i \in I^{\bar{t}}$*

$$\mu^{\bar{t}}(i) R_i^{\bar{t}} \mu^t(i) \quad \forall t < \bar{t}$$

It worths point out that keeping a promise does not preclude the case where the student ends up matched to himself while a school were proposed at some early stage. For instance, if $\mu^t(i)$ is a school proposed at date t that is not acceptable at date $\bar{t}$ according to $P_i^{\bar{t}}$ then it might be that $\mu^{\bar{t}}(i) = i$. But, strictly speaking, the promise of "being better matched" is still kept.

4.2. The case of fixed preferences

In this section we consider the polar case of no refitting across stages. From the viewpoint of students it means that once the have submitted $(P_i^1)_i$, the sole degree of freedom left to students in the multistage mechanism is the decision of acceptance or rejection at each stage. We obtain contrasted results depending on the choice of the one-stage matching mechanisms (either SOSM or COSM).

We have in mind two distinct definitions when we mean fixed preferences. Under assumption A (which almost delimits the stability of $\mathcal{M}^{COSM}$), fixed preferences means the truncation mapping as defined in Remark 1. Under assumption B (which delimits the stability for $\mathcal{M}^{SOSM}$), fixed preferences means the identity mapping as defined in Remark 2.

[17] As far as the term "better" is correctly interpreted by students since it refers to their future preferences to be submitted.

Proposition 3 *Suppose that F is the truncation mapping. Under $\mathcal{M}^{COSM}$ the students obtain the same outcome as in COSM for the initial problem: $\nu^{COSM}(\Gamma, \widehat{\succ}_I) = COSM(\Gamma)$ for any school choice problem $\Gamma = (P^1, I^1, q^1)$. In addition, the multistage mechanism $\mathcal{M}^{COSM}$ is stable.*

Proof At stage 2 of the multistage mechanism $\mathcal{M}^{COSM}$ consider the following matching $\tilde{\mu}^2$ for the problem (P^2, I^2, q^2): $\tilde{\mu}^2(i) = \mu^1(i)$ for every $i \in I^2$. That is, every student who declines the proposal at stage 1 is proposed the same assignment. We first note that $\tilde{\mu}^2$ is individually rational. It comes from the individual rationality of μ^1 and from the truncation mapping that generates the students' preferences P^2. Second it is an easy matter to check that there is no blocking pair at μ^2. The intuition is that there is less competition at stage 2 than at stage 1, hence the absence of blocking at stage is maintained at stage 2.

The matching μ^2 is thus stable and is such that every student i is assigned either to himself or to the least preferred school with respect to P_i^2 (recall that A-refitting holds). It follows that the matching μ^2 is the least preferred matching for the students in the set of stable matchings for the problem (P^2, I^2, q^2). Suppose not, then the least preferred stable matching must necessarily assign less seats than in μ^2. That is, one student remains unmatched but prefers a school where there is a vacant seat. This contradicts the non-wastefulness of that stable matching.

We have just proved that $\tilde{\mu}^2$ is $COSM(P^2, I^2, q^2)$. Repeating the arguments for the next steps, we obtain that students are assigned to the same seats across the stages until they accept the proposals. Hence $\nu^{COSM}(\Gamma, \widehat{\succ}_I)(i) = \mu^{t_i}(i) = \mu^1(i) = COSM(\Gamma)(i)$ by definition.

The second statement follows directly from the first one since $COSM(\Gamma)$ is stable for the problem Γ. ■

For the multistage version with SOSM we obtain different conclusions when students follow the identity or truncation mapping in the resubmission phase.

Proposition 4 *Suppose that F is the identity (or truncation) rule.*

The outcome of $\mathcal{M}^{SOSM}$ is weakly preferred to the SOSM outcome according to the initial preferences: $\nu^{SOSM}(\Gamma, \widehat{\succ}_I) R_i^1 \; SOSM(\Gamma)$ for every student $i \in I^1$ and any school choice problem $\Gamma = (P^1, I^1, q^1)$ (with a strict preference in some instances).

In addition, the multistage mechanism $\mathcal{M}^{SOSM}$ is stable over early matches but not stable in general.

$P_{i_1}^1 = P_{i_1}^2$	$P_{i_2}^1 = P_{i^2}^2$	$P_{i_3}^1$	P_{s_1}	P_{s_2}	P_{s_3}
s_2	s_1	s_1	i_1	i_2	i_3
s_1	s_2	s_3	i_3	i_1	$\vdots$
			i_2	$\vdots$	

Table 2: SOSM not equivalent to $\mathcal{M}^{SOSM}$ with fixed preferences

Proof Since the identity mapping is a B-refitting rule one can use Proposition 2. Thus it holds that $\mu^{t_i}(i) R_i^{t_i} \mu^1(i)$ by choosing $\bar{t} = t_i$ and $t = 1$. By definition $\nu^{SOSM}(\Gamma, \widehat{\succ}_I)(i) = \mu^{t_i}(i)$ and $\mu^1(i) = SOSM(\Gamma)$. To prove the first statement of the proposition it remains to find an instance of school choice problem where the preference is strict. This is given in Table 2, which describes a school choice problem with three students, three schools, one seat in each school, and $T = 2$.

Here $SOSM(\Gamma)$ is given by the three pairs (i_1, s_1), (i_2, s_2) and (i_3, s_3). Clearly if i_3 accepts at stage 1 while i_1, and i_2 declines and resubmits the same preferences. The final match of the multistage matching mechanism $\mathcal{M}^{SOSM}$ is (i_1, s_2), (i_2, s_1) and (i_3, s_3), as it is desired.

Using the same example, one checks that it is an instance where the final outcome of $\mathcal{M}^{SOSM}$ is not stable with respect to the initial preferences. Finally, stability over early matches is deduced from Theorem 2 since the identity mapping is a B-refitting rule. This proves the second statement of Proposition 4.

Note that the truncation mapping is also B-refitting rule and that the arguments in the example still go through under truncation. Hence the statement of Proposition 4 holds for the truncation mapping. ∎

Some general comments are in order. Proposition 3 shows that using the multistage matching mechanism $\mathcal{M}^{COSM}$ makes sense only if one allows for some degree of refitting. If the preferences are kept fixed, students have no particular interest to delay their decisions since the final assignments are fully determined before the first stage when they submit their preferences. This feature is specific to fixed preferences. If it is not the case then the students may benefit from delaying their decision of acceptance. For instance, if one good student (is forced to declare) declares all schools as being unacceptable after some stage t, some students may be better off, all things being equal.

In the case of $\mathcal{M}^{SOSM}$ the interpretation is different. Proposition 4 shows that even if preferences are fixed, it might be tempting for sufficiently patient students to delay their acceptance since by doing so they may obtain a better match. Overall, $\mathcal{M}^{SOSM}$ Pareto improves SOSM under fixed preferences. Not surprisingly, the cost to pay for such an improvement is the stability.[18] In Section 5, we build on these feature to deduce equilibrium strategies for the students playing $\mathcal{M}^{SOSM}$ with the identity mapping.

4.3. Dealing with dismissals

A rationale for using multistage matching mechanisms is to handle dismissals during the matching procedure. We incorporate here the possibility that students dismiss in a subsequent stage and leave vacant the seats they accepted earlier. In practice, dismissals can occur for candidates involved in overlapping centralized matching mechanisms (charter schools vs public schools) or for candidates who fail the college entrance examination as in the French college admissions (see Section 6). For the case of multiple school systems, Manjunath and Turhan (2016) defines a new assignment process to deal with the vacant seats generated by students who dismiss from one of the two systems. The proposed mechanism, which is also sequential, shares some similarities with the multistage matching mechanisms.[19]

Dismissals can be easily accommodated by using a new specification of the multistage matching mechanisms. The general structure of the multistage matching mechanism is unchanged but we allow the candidates already assigned to a school to notify a dismissal at every intermediary stage.

Let θ be any one-stage matching mechanism for school choice problems.

The input is a school choice problem $\Gamma = (P^1, I^1, q^1)$. Let ν be the null matching for Γ.

- Stage t, $1 \leq t < T$. Set $\mu^t := \theta(P^t, I^t, q^t)$.

  For every $i \in I^t$:

  Accept. : *as before*

  Decline. : *as before*

  Dismissal. Every student $i \in I^1 \setminus I^t$ such that $\nu(i) \in S$ can notify a dismissal, then:

[18] We disregard so far the issue of time-dependent preferences, see Section 5.

[19] See also Ekmekci and Yenmez (2015). They analyze the presence of multiple school systems from the viewpoint of the schools.

– set $P_i^t := [i, \ldots];$

– $\nu(i) := i;$

– $t'_i := t;$

– $q^{t+1}_{\nu(i)} := q^t_{\nu(i)} + 1.$

- Stage T. *as before*[20]

The submitted preferences of student i in the above procedure are now either a list $(P_i^1, P_i^2, \ldots, P_i^{t_i})$ or a list $(P_i^1, P_i^2, \ldots, P_i^{t_i}, \emptyset, \ldots, \emptyset, P_i^{t'_i})$, where $P_i^{t'_i} = [i, \ldots]$ if student i dismisses. We maintain however the notation $\widehat{\succ}_{I^1}$ for the paths of submitted preferences for all students since no confusion arises.

Overall, the occurrence of a dismissal does not affect our previous results on stability. The intuition is that our main solution concept of stability over early matches is backward looking. Hence a dismissal makes no change for those who have accepted earlier a proposal. For the remaining students, the dismissal offers a new vacant seat that can be assigned if needed. Obviously, dismissing students cannot block the final match by construction. Hence stability with early matches is maintained under the same conditions. Of course, the possibility of dismissals has more implications on the strategic side. Intuitively it provides incentives to delay acceptance but also to manipulate the preferences (see Section 5).

Let $\mathcal{M}_D^\theta$ denote the multistage matching mechanism with dismissals using θ as one-stage matching mechanism. The underscript D is used as well to denote the resulting outcomes of such mechanisms.

> *If dismissals are allowed, Theorems 1 and 2, Propositions 1, 2, 4 can be readily adapted by replacing $\mathcal{M}^{(\cdot)}$ by $\mathcal{M}_D^{(\cdot)}$.*

Proposition 3, dealing with fixed preferences under COSM, is the result with no exact counterpart. Clearly one cannot guarantee the assignments to be unchanged across stages if some students dismiss and leave behind vacant seats. However it is easy to check that the statement of Proposition 4 can be restated to handle COSM as well.

[20] There is no dismissal in the last stage.

Proposition 5 *Suppose that F is the truncation mapping. The outcome of $\mathcal{M}_D^{COSM}$ is weakly preferred to the COSM outcome according to the initial preferences:*

$$\nu_D^{COSM}(\Gamma, \widehat{\succ}_I) R_i^1 \, COSM(\Gamma)$$

for every student $i \in I^1$ and any school choice problem $\Gamma = (P^1, I^1, q^1)$ (with a strict preference in some instances). In addition, the multistage mechanism $\mathcal{M}_D^{COSM}$ is stable over early matches but not stable in general.

For sake of completeness it worths pointing out the following obvious feature. Loosely speaking, students who do not dismiss can benefit from dismissals in $\mathcal{M}^\theta$ if θ is either SOSM or COSM. The proof simply relies on the fact that allowing for dismissals tends to increase the number of seats available at every stage.

Proposition 6 *Let θ be either COSM or SOSM. If F be the identity mapping then, all things being equal, every student $i \in I^t$ obtains a weakly better proposal at stage t whenever some dismissals have been notified at stage $t-1$.*

Proof Let (I^t, P^t, q^t) be the school choice problem that is defined at stage $t-1$ after the resubmission of preferences but before the notification of dismissals. In case of no dismissal, the school choice is used at stage t to compute the proposal $\theta(I^t, P^t, q^t(i))$ made to student i. In case of notifications, the new problem is (I^t, P^t, q_D^t) where q_D^t is the new vector of capacities satisfying $q_D^t \geq q^t$. If one considers the two extreme matchings SOSM and COSM it is well known that we have both:

$$SOSM(I^t, P^t, q_D^t)(i) R_i^t \; SOSM(I^t, P^t, q^t)(i)$$

and

$$COSM(I^t, P^t, q_D^t)(i) R_i^t \; COSM(I^t, P^t, q^t)(i)$$

■

4.4. Dealing with scheduling constraints

As emphasized in the introduction, the class of multistage matching mechanisms can accommodate the scheduling constraints of the candidates. Some are forced to get their match early while other are able to decide at the end of the procedure. We formalize here precisely this idea.

The scheduling constraints are summarized by a $|I| \times |S|$ - matrix K. For every pair (i, s) we define $K(i, s) \in \{1, \ldots, T\}$ as being the stage until which the school s is *feasible* for student i. It means that, beyond this stage, the school is not acceptable according to the student's preferences. For every t, let K_i^{t+1} be the set of preferences that are compatible with feasible schools of student i at date $t+1$ (irrespective from F):

$$K_i^{t+1} := \{P_i \in \mathcal{P}_i \, : \, iP_is \text{ for every school } s \text{ s.t. } K(i,s) < t+1\}$$

To accommodate those constraints the students can declare them either explicitly or via the refitting rule F. We follow the latter option. In the next definition where, at every stage, the students can submit a preference that is consistent with F but also with the constraints K.

Definition 5 A refitting rule F is schedule-preserving if, for every student i, $F(P_i, p) \cap K_i^{t+1} \neq \emptyset$ for every $(P_i, p) \in \mathcal{P}_i \times (S \cup \{i\})$ and every K.

Clearly any single valued refitting rule does not satisfy the property of schedule-preserving. It is an easy matter to characterize the set of schedule-preserving rules. The rule must be such that, for every subset of schools A and every past preferences, there exists a preference consistent with respect to the rule F where only schools in A are acceptable. It is formalized in the next proposition, whose proof is straightforward.

Proposition 7 *A refitting rule F is schedule-preserving if and only if for every $A \subset S$ and $(P_i, p) \in \mathcal{P}_i \times (S \cup \{i\})$ there exists $P'_i \in F(P_i, p)$ such that $sP'_i i$ implies $s \in A$.*

Proof The proof is immediate since the assertion is merely a modification in the phrasing of the definition. First, for any given set K, if A is the set of schools such that $K(i,s) \geq t+1$ then the element P'_i in $F(P_i, p)$ such that $sP'_i i$ implies $s \in A$ satisfies $P'_i \in F(P_i, p) \cap K_i^{t+1}$. Second, for any given set $A \subset S$, if K is such that $K(i, s) = 1$ for every $s \notin A$ then $P'_i \in F(P_i, p) \cap K_i^{t+1}$ is such that $sP'_i i$ implies $s \in A$. ∎

Remark 6 There exist schedule-preserving refitting rules satisfying Assumption A or B.

5. Strategic issues

Is it sufficient to consider one-stage strategyproof mechanisms (like SOSM) to guarantee that the same property holds at the multistage level? The answer is clearly no in general.

We first illustrate the possibility of manipulation through a simple example. We formalize then more precisely the strategic problem as an extensive game with simultaneous moves. We show how a natural truthful equilibrium can emerge nevertheless in some specific environments. But the results are fragile and collapse as soon as scheduling constraints appear in the students' preferences, which can be viewed as a first pass towards the modeling of intertemporal preferences.

The positive results hold when SOSM is chosen as the one-stage mechanism. Obviously the case of COSM can be right away ruled out since the property of strategyproofness is not even satisfied at the one-stage level.

5.1. Strategyproofness: a counter-example

The example features an other-delaying manipulation. The school choice problem has three schools, s_1, s_2, and s_3, and three students, i_1, i_2, i_3. Each school has one seat to offer. Let the students' true preferences (constant over stages and without discounting) and schools' priorities be as follows

$\widetilde{P}_{i_1}$	$\widetilde{P}_{i_2}$	$\widetilde{P}_{i_3}$	P_{s_1}	P_{s_2}	P_{s_3}
s_1	s_2	s_2	i_2	i_1	i_1
s_2	s_1	s_3	i_3	i_2	i_2
s_3	s_3		i_1	i_3	i_3

[N.B. $s_2\widetilde{P}_{i_2}s_1$ means here that s_2 is always preferred to s_1 whatever the stage at which the seat is obtained.]

Consider that the mechanism $\mathcal{M}^{SOSM}$ is used to allocate seat to students.

We assume that students i_1 and i_2 both follow the same strategy. They submit their true preferences in the first stage. If they are proposed their top choice then they accept the proposal, otherwise they decline the proposal and declare every school as being unacceptable in the second stage (consistent with B-refitting).

If student i_3 submits his true preference in the first stage then i_1 is now matched to s_1, i_2 matched to s_2 and i_3 matched to s_3. The students i_1 and i_2 accept. Accepting a seat in

s_3 is then the best outcome for student i_3.

If student i_3 submits $P^1_{i_3} := [s_2, s_1, s_3]$ then the output of SOSM in the first stage is $\mu^1 = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$. Students i_1 and i_2 decline the proposals and submit respectively $P^2_{i_1} = [i_1]$ and $P^2_{i_2} = [i_2]$. In that case, if student i_3 declines the proposal s_3 and submits again $P^2_{i_3} := [s_2, s_1, s_3]$ then he obtain eventually a seat in s_2 at the end of stage 2. The manipulation is profitable.

The main intuition is straightforward. Manipulating cannot make the student i_3 better off in the first stage (from the strategyproofness of SOSM). However by doing so student i_3 modifies the assignments of other students at stage 1, which can be beneficial in the next stage.[21]

5.2. *A general framework*

The possibility for students to submit new preferences and to decline proposals at every stage requires to model the problem as an extensive game.

Given a multistage matching mechanism $\mathcal{M}^\theta$, one can associate an *extensive game with perfect information and simultaneous moves* $\mathcal{E}(\mathcal{M}^\theta)$. At each stage t, (remaining) students play a simultaneous (quasi) preference revelation game. Formally, at the beginning of the game, i.e. with the empty story, every student i chooses

$$a_i(\emptyset) \in A_i(\emptyset) = \mathcal{P}_i$$

Let h be any non terminal history, which starts at stage $t < T$, then h specifies in particular the problem $\Gamma^t = (P^t, I^t, q^t)$ and the matching μ^t. Any student $i \in I^t$ chooses an action

$$a_i(h) \in A_i(h) = \{Y, N\} \times F(P^t_i, \mu^t(i))$$

If $a_{i1}(h) = Y$ then the final payoff of student i is $\nu(i) = \mu^t(i)$; the profile of actions (a^t_i, a^t_{-i}) generates a new problem $\Gamma^{t+1} = (P^{t+1}, I^{t+1}, q^{t+1})$ which in turn generates the outcome of the simultaneous game $\mu^{t+1} = \theta(\Gamma^{t+1})$; at the beginning of stage $t+1$, the history $(h, (a_i, a_{-i}))$ specifies thus the pair $(\Gamma^{t+1}, \mu^{t+1})$. If $t+1 = T$ the outcome is $\nu(i) = \theta(\Gamma^T)(i)$ for every remaining student $i \in I^T$.

[21] In the example, note however that students i_1 and i_2 play weakly dominated strategies as long as true preferences are assumed to be fixed across stages. Especially they both decline a seat at stage 1 and obtain nothing at stage 2.

A strategy of student i maps the set of non terminal histories h for which i belongs to the set of remaining players to the set $A_i(h)$. The solution concept is the subgame perfect equilibrium (SPE).

We maintain a restrictive assumption on students' preferences in the game $\mathcal{E}(\mathcal{M}^\theta)$. For every student i and terminal history h, the preferences $(\succ_i)_i$ of students over histories are characterized by a profile $(P_i)_i$ of (static) preferences over schools (P_i is a linear ordering over $S \cup \{i\}$). That is, for every i and terminal histories h, h', it holds that

$$h \succ_i h' \text{ if and only if } \nu(i)_{|h} P_i \nu(i)_{|h'}$$

where $\nu(i)_{|h}$ reads the payoff of student i when the students follows the precepts given by h. The assumption is restrictive since it excludes stage-dependent preferences or preferences with discount factors. In particular student i evaluates the outcome of the game with respect to the school he obtains irrespective of the date he accepted the seat (i.e., when he declares (Y)es).

Even within that restrictive framework, the example of Section 4.1 shows that students can strategize across stages. In particular, there is no dominant equilibrium in the game $\mathcal{E}(\mathcal{M}^{SOSM})$ in general.

5.3. Subgame perfect equilibrium and truthful strategies

In that section, we make no restriction on the refitting rule F used in the multistage matching mechanism. That is, F is the set of all possible orderings over $S \cup \{o\}$. Despite the non existence of a truthful dominant strategy (see Section 4.1) we can easily identifies a simple equilibrium of the game $\mathcal{E}(\mathcal{M}^\theta)$ where students behave sincerely.

Proposition 8 *Let $\Gamma = (I, S, (P_s, q_s)_{s\in S}, (\tilde{P}_i)_{i\in I})$ be a school choice problem. The profile of strategies where every student i declines any proposal until the last stage and submits the true preference $\widetilde{P}_i$ at each stage is a subgame perfect equilibrium of the game $\mathcal{E}(\mathcal{M}^{SOSM})$.*

Proof Let $f(i)$ be the outcome for student i when the students follow the strategies given in the statement of the proposition. First, we remark that any deviation in the preferences submitted by student i has no consequences on the uniform strategies of the other students. Thus to assess whether it is profitable for student i to deviate from his strategy it suffices to consider a deviation from $\tilde{P}_i$ at some stage t, followed by an acceptance in the stage $t+1$.

P_{i_1}	P_{i_2}	P_{i_3}	P_{s_1}	P_{s_2}	P_{s_3}
s_1	s_2	s_1	i_2	i_1	i_3
s_2	s_1	s_2	i_3	i_3	
		s_3	i_1	i_2	

Table 3: No equilibrium with differentiated acceptance dates

From the strategyproofness of SOSM, his best option is to declare $\tilde{P}_i$ at t. Since the other students always declare their true preferences, student i obtains $f(i)$ at stage $t+1$ instead of stage T. From our assumption on preferences the student is indifferent between those two outcomes. Thus there is no profitable deviation. ■

The proof of the previous result can be readily adapted to obtain a more general statement.

Proposition 9 *Let* $\Gamma = (I, S, (P_s, q_s)_{s\in S}, (\tilde{P}_i)_{i\in I})$ *be a school choice problem. The profile of strategies where every student* i *submits the true preference* $\widetilde{P}_i$ *at each stage and all students accept the proposal at the same stage is a subgame perfect equilibrium of the game* $\mathcal{E}(\mathcal{M}^{SOSM})$.

The result cannot extend to strategies with differentiated acceptance dates.

Example 2 In the problem given in Table 3, SOSM mechanism provides the matching $\mu = ((i_1, s_2), (i_2, s_1), (i_3, s_3))$ if the students submit their true preferences. In the extensive game $\mathcal{E}(\mathcal{M}^{SOSM})$ with two stages, if the students always submit their true preference and accept the proposal at stage 2 for student i_1 and at stage 1 for students i_2 and i_3 then the outcome is again μ. However it is not a subgame perfect equilibrium. If student i_2 can deviate by declining the offer at stage 1, then he obtains at stage 2 a seat at school s_2 since the SOSM takes place for a problem with students i_1 and i_2 only (and the same for student i_1, he gets a better seat at school s_1).

As we have seen in the previous section, the use of specific refitting rules provides some structure on the payoffs (e.g. Proposition 2). Unfortunately, it does not bring new insights on the equilibria of the strategic game $\mathcal{E}(\mathcal{M}^{SOSM})$.

We start with the polar case of a refitting rule with unchanged preferences across stages, i.e. the identity mapping. Students submit once their preferences at the beginning, and

P_{i_1}	P_{i_2}	P_{i_3}		P_{s_1}	P_{s_2}	P_{s_3}
s_1	s_2	s_1		i_2	i_1	i_3
	s_1	s_2		i_3	i_3	
		s_3		i_1	i_2	

Table 4: No dominant sincere strategy with the identity mapping

choose at which stage they accept the proposal, which can differs from stage to stage. With respect to the extensive game $\mathcal{E}(\mathcal{M}^{\theta})$, it can be easily modeled by assuming that F is the identity mapping (see Remark 2 for the definition of the rule).

Alternatively, we remark that assuming $F = Id$ in $\mathcal{E}(\mathcal{M}^{\theta})$ is equivalent to an extensive game denoted by $\mathcal{E}(\mathcal{M}^{\theta}_{Id.})$ where every student i has the following actions sets $A_i(.)$ at every non terminal history:

$$A_i(\emptyset) = \mathcal{P}_i \text{ and } A_i(h) = \{Y, N\} \text{ for every other } h$$

First we note that the example of Section 5.1 does not apply to that framework since the students were able to modify the preference they submitted at every stage. But surprisingly, the same feature on preference manipulation continues to hold even if students are enforced to keep up with their very first submitted preferences. It is illustrated in the next example.

Example 3 The problem is similar to the one given in Example 2 except for student i_1 who finds that only school s_1 is acceptable (see Table 4). In that case the student i_1 is able to manipulate the outcome of the multistage matching mechanism for some specific instances of strategies of the other players. First, we note that SOSM with the true preference gives the matching $(i_1, i_1), (i_1, s_2), (i_3, s_1)$. So whatever being the acceptance dates, the student i_1 will never get a seat in $\mathcal{E}(\mathcal{M}^{SOSM}_{Id.})$ if what the students submit agree with their true preferences.

Assume that i_3 always accepts its first proposal and that i_2 always declines in the first stage. Consider the following strategy for i_1: he submits $P'_{i_1} = [s_1, s_2, i_1]$ and declines in the first stage whatever he gets. In the first stage, SOSM is $\mu' = ((i_1, s_2), (i_2, s_1), (i_3, s_3))$. Student i_3 accepts s_3 and in the second stage student i_1 obtains the seat in s_1 and student 2 obtains the seat in s_2. Hence, by using P'_{i_1} gets his first choice instead of remaining alone by submitting his true preference P_{i_1}

Proposition 8 and 9 still hold true if one assumes that F is the identity mapping or satisfies B-refitting. It suffices to note that the strategy consisting in submitting the true preferences at every stage is compatible with F.

To conclude, one can easily define a natural extension of the game $\mathcal{E}(\mathcal{M}^\theta)$ to account explicitly for heterogenous scheduling constraints. The extensive game with constraints K, denoted by $\mathcal{E}^K(\mathcal{M}^\theta)$ is defined as before except for the preferences which are stage-dependent.

If $\tilde{P}_i$ is the true preference of student i then we denote $\tilde{P}_{i\,|\,K_i^t}$ the truthful preferences compatible with K at stage t. It can be viewed as the projection of $\tilde{P}_i$ on the set K_i^t of compatible preferences at stage t. Formally, $\tilde{P}_{i\,|\,K_i^t}$ satisfies $\tilde{P}_{i\,|\,K_i^t} \in K_i^t$ and

$$s\tilde{P}_{i\,|\,K_i^t}s \text{ iff } s'\tilde{P}_i s \;\; \forall s, s' \text{ s.t. } K(i,s) > t, K(i,s') > t, \; s\tilde{P}_i i, \; s'\tilde{P}_i i$$

[the ordering $\tilde{P}_{i\,|\,K_i^t}$ is not uniquely defined for the non admissible schools, but it is innocuous for our matter.]

Even in the most specific case of the identity refitting rule, there is room for manipulation. To see that, one can use Example 3. Consider that students i_1 and i_2 have no constraints while student i_3 needs to accept any proposal in the first stage. Formally, $K(i_1, s) = K(i_2, s) = 2$ and $K(i_3, s) = 1$ for all s. Student i_1 can manipulate his preferences to get his top choice at the expense of student i_3.

In other words the analogue to Proposition 9 does not hold if the students have heterogenous but uniform scheduling constraints as in the example. Submitting the true preferences until the last feasible stage is not a subgame perfect equilibrium of $\mathcal{E}^K(\mathcal{M}^{SOSM})$, even if one assumes $F = Id$.

Remark 7 A game-theoretic analysis that account for possible dismissals as in Section 4.3 is omitted since it goes far beyond the scope of the paper. An adapted setting to deal with the problem seems to be an incomplete information framework where each student has a belief on the possibility to dismiss at some stage.

6. Example from the field: the French college admissions

The French online application "Admission Post-Bac" is an hybrid large scale two-sided matching mechanism populated by selective and non-selective colleges, and candidates who can be assigned sequentially to colleges and update their preferences across time. Does APB succeed to secure basic properties of a matching mechanism?

Each year over 500,000 French high-school students connect with the online APB application to enter college programs. Students submit their strict preference over colleges that are constrained by capacities and either selective (IUT, prepas) or non selective (universities). The APB procedure runs a DA algorithm (the college proposing version...) to deal with the match of selective colleges, while APB is based on a random matching procedure for the non selective colleges.

Besides the joint mixture of two mechanisms (DA and random matching), the originality of APB comes from the **3-round** mechanism that takes place from June to July. Roughly, the 3-round procedure allows students to accept or reject the offer they receive (if any) and to update their preferences, at each round.

An interesting feature is that APB is able to make early promises, say in June, before knowing the final admissible pool of students, which is a subset of the candidates initially registered. Indeed, around 20% of the candidates will not qualify for the national exam to enter colleges (the baccalaureat), and others will enter post high-school selective programs that are not handled by APB (U. Paris-Dauphine, Sciences-Po, nursery schools). Those candidates are not yet known in June, though they are in the last stage. They enter the pool of APB candidates and temporarily claim seats during the first rounds.

The final third round takes place in mid-July when the admissible pool of students is completely known (results of the high-school national exam, etc...) and their preferences as well. Why not matching everybody at this time then? We suspect that APB is precisely designed to speed-up the allocation of students, which can be both a political objective of the Ministry and helpful instrument for students, for campus housing issues for instance. Actually, a stylized fact of APB is that 80% of the student body who will enter a college by September is already allocated by June (data: Ministry of Education 2009)![22] A natural question to ask is thus whether the final match guarantees stability. A more precise look at the proposed APB match enables us to answer in the affirmative (at least if one considers the selective colleges only).

In 2015, the APB schedule is as follows: Round 1: June 8th – June 13th; Round 2: June 25th – June 30th; Round 3: July 14th – July 19th. At each round, the students still in competition will choose among the following:

- OUI: you accept definitely the proposal and you are removed from the pool of candi-

[22] That is, 80% of the students who pass the exam and do not exert their outside option.

dates. Every other college is removed from your vows. You are assigned the proposal unless you dismiss from APB in the next steps.[23]

- OUI MAIS (not available in the last round): you accept temporarily the proposal, but you wish to remain in the pool of candidates to obtain a college at least as good as the proposal (as long as the latter remains acceptable in your preferences).
- NON MAIS (not available in the last round): you decline the proposal, which is removed definitely from your preference list, but you wish to remain in the pool of candidates in the next stages. You might end up with no college.
- NON: you dismiss from APB.
- (UPDATING PREFERENCES) In addition, students can drop at the same time the proposed college or some of the colleges with higher priority (if any) from their lists of preferences. Colleges with lower priority than the proposed college are automatically removed from their preference lists.[24]

Importantly, we will show that each of the four answers (oui, oui mais, non mais, non) can be readily encapsulated into an updating rule. We use this feature to simplify the presentation of the APB model in the remainder.

In the following, we consider only selective colleges who submit strict priorities over students. We start by modeling first the APB refitting rule.

Definition 6 The refitting rule F satisfies assumption C (C-refitting) if, for every student $i \in I$, any preference $P_i \in \mathcal{P}_i$ and proposal $pR_i i$, it holds that for all $\widetilde{P}_i \in F(P_i, p)$

$$s\widetilde{P}_i s' \;\Leftrightarrow\; sP_i s', \;\forall s, s' \text{ s.t. } s\widetilde{P}_i i \text{ and } s'\widetilde{P}_i i$$

Remark 8 Note that Assumption C implies B. In addition, the identity mapping and the truncation mapping defined respectively in Remarks 2 and 1 satisfies Assumption C.

Claim 1 *The APB refitting rule F^{APB} is the largest rule (for set inclusion) satisfying A and C-refitting.*

[23] If the candidate fails at the national exam for instance.

[24] Of course, updating and answers must be mutually consistent (for instance, one cannot say OUI MAIS and remove the proposed college from vows). Updating might be also redundant with their choices. For example, OUI is equivalent to declare every college ranked above their proposed college as non admissible.

We are now in position to describe the APB in terms of our previous mechanism. To do so note that, at each stage t, the "oui" students are those who accept the proposal; the "oui mais" students are those who decline a proposal $\mu^t(i)$ and update their preferences in such a way that $\mu^t(i)P_i^{t+1}i$; the "non mais" students are those who decline a proposed school $\mu^t(i)$ and update their preferences in such a way that $iP_i^{t+1}\mu^t(i)$; the "non" students are those such that $iP_i^{t+1}s$ for every s.[25]

Claim 2 *The APB procedure is $\mathcal{M}^{COSM}$ with dismissals and $T = 3$.*

We deduce then the main positive feature of the French college admissions.

Theorem 3 *The French college admission system (for selective colleges only) is stable over early matches with a schedule-preserving refitting rule.*

Recall that the case of non-selective colleges (here the French universities) goes beyond the scope of our paper. A full understanding of the APB procedure requires however to consider both types of colleges simultaneously since they appear together in the single lists of colleges submitted by students. It is likely that the statement of Theorem 3 will not survive in the general setting.

7. Appendix

7.1. Proof of Theorem 1

Suppose by way of contradiction that stability with early matches is not satisfied. By construction and since $\mathcal{M}^{COSM}$ produces a stable matching μ^t at each stage, the first two conditions of Definition 1 are both satisfied. Hence, there is a student i with justified envy against an early pair (j, s). Without loss of generality, we assume next that $t_i = T$. Since the matching μ^T is stable under COSM, the student j is necessarily assigned to s at a preliminary stage. This can be stated as a lemma.

Lemma 1 *There exists j such that $\nu(j) = s$, iP_sj, and $sP_i^T\nu(i)$ with $t(j) < T$ and $t(i) = T$.*

[25] The students who do not get anything in the first round ($\mu^t(i) = i$) but still wish to go on (at least one admissible school in P_i^{t+1}) enter in the oui mais category.

Since i has been matched only at date T, i has always updated his preferences according to F_i. Let $P_i^{t(j)}, P_i^{t(j)+1}, \ldots, P_i^T$ be the list of preferences considered respectively at the beginning of stage $t(j), t(j)+1, \ldots, T$ satisfying

$$P_i^{t(j)} \in F_i\big(P_i^{t(j)-1}, \mu^{t(j)-1}(i)\big), P_i^{t(j)+1} \in F_i\big(P_i^{t(j)}, \mu^{t(j)}(i)\big), \ldots, P_i^T \in F_i\big(P_i^{T-1}, \mu^{T-1}(i)\big).$$

We need the following lemma.

Lemma 2 *At every date $t(j) < t \leq T$, it holds that $\mu^t(i) R_i^t \mu^{t-1}(i)$ for every $i \in I^t$.*

Proof Suppose then by way of contradiction that $\mu^{t-1}(i) P_i^t \mu^t(i)$. Note that $\mu^{t-1}(i)$ is thus necessarily a school. Since $\mu^t \in \Sigma(P^t, I^t, q^t)$ there exists necessarily a student i^1 such that $\mu^t(i^1) = \mu^{t-1}(i)$ and $\mu^{t-1}(i^1) \neq \mu^{t-1}(i)$ with $i^1 P_{\mu^{t-1}(i)} i$. Since $\mu^t(i^1)$ is a school, it must be that $\mu^t(i^1) P_{i^1}^{t-1} \mu^{t-1}(i^1)$ since schools s such that $\mu^{t-1}(i^1) P_{i^1}^{t-1} s$ are removed from the preferences in the next period, under A-refitting. We obtain thus $\mu^{t-1}(i) P_{i^1}^{t-1} \mu^{t-1}(i^1)$. But this contradicts the stability of μ^{t-1} where i is proposed $\mu^{t-1}(i)$ since $(i^1, \mu^{t-1}(i))$ is then a blocking pair. ■

We have just shown that

$$\mu^T(i) R_i^T \mu^{T-1}(i) R_i^{T-1} \ldots \mu^{t(j)+1}(i) R_i^{t(j)+1} \mu^{t(j)}(i).$$

Using the fact that $\mu^T \in \Sigma(P^T, I^T, q^T)$ is individually rational, it holds that $s P_i^T i$ since $s P_i^T \nu(i) = \mu^T(i)$. From A-refitting one also deduces that $s P_i^k i$ at any previous step $k < T$.

Recall that $i P_s j$, where $s = \mu^{t(j)}(j)$ from Lemma 1. Since $\mu^{t(j)} \in \Sigma(P^{t(j)}, I^{t(j)}, q^{t(j)})$ it must be necessarily the case that $\mu^{t(j)}(i) R_i^{t(j)} s$, which implies in turn that $\mu^{t(j)}(i) = s$ under A-refitting.

We have obtained the following features: i obtains s and declines the proposal at the stage t; the school s appears in the vows of student i at every subsequent stage. Under A-refitting, this means necessarily that s is the school that appears at the bottom of the submitted preferences and that is proposed at every stage of the mechanism. So $\mu^T(i) = \mu^{T-1}(i) \ldots \mu^{t(j)}(i) = s$. But this contradicts $s P_i^T \nu(i)$ obtained in Lemma 1.

7.2. Proof of Theorem 2

(1) ⇒ (2). The proof is similar to the one of Theorem 1 except for the analogue of Lemma 2 in which the interplay between the assumption B and SOSM requires an entire new proof.

Suppose by way of contradiction that stability with early matches is not satisfied. That is, (wlog) there is exists i such that $t_i = T$ who has a justified envy over an early pair (j, s). Notice that Lemma 1 still holds true here.

Lemma 3 *For each $t \leq T$, and each $i \in I^t$, $\mu^t R_i^t \mu^{t-1}$.*

Proof For each school s, let $U_s = \{i \in I^t : \mu^t(i) = s \text{ and } \mu^{t-1}(i) P_i^t s\}$ and $V_s = \{i \in I^t : \mu^{t-1}(i) = s \text{ and } s P_i^t \mu^t(i)\}$. If the statement of the Lemma does not hold true, then $U_s \neq \emptyset$ for some s.

We claim that for any s such that $U_s \neq \emptyset$ we have $|U_s| = |V_s|$. Note that $U_s \cap V_s = \emptyset$. Also, for each $i \in U_s$, since μ^t is stable at t we have $j P_s i$ for each $j \in V_s$. Observe that if $i \in U_s$ for some school s, μ^t being individually rational at t and preferences being strict implies $\mu^{t-1}(i) \in S$. So, $i \in V_{\mu^{t-1}(i)}$. Therefore, we have $\cup_s U_s = \cup_s V_s$. Also, since $U_s \cap U_{s'} = \emptyset$ for any two schools $s \neq s'$, to prove the claim it suffices to show that for any s we have $|V_s| \leq |U_s|$.

Suppose then that for some school s we have $|V_s| > |U_s|$. From the construction of V_s, there are $|V_s|$ students who lost their seats at s, and prefer s to their new schools at t. Hence, at time t, there are necessarily at least $|V_s|$ newcomers at school s. If $|V_s| > |U_s|$ there is a set W_s of students not in U_s such that $|W_s| = |V_s| - |U_s|$ and for each $i \in W_s$, $\mu^{t-1}(i) \neq \mu^t(i) = s$. Since $W_s \cap U_s = \emptyset$ we have $s P_i^t \mu^{t-1}(i)$ for each $i \in W_s$. From the stability of μ^t at t we also have $i P_s j$ for each $i \in W_s$ and each $j \in V_s$. Also, from assumption B, $s P_i^t \mu^{t-1}(i)$ implies $s P_i^{t-1} \mu^{t-1}(i)$ for each $i \in W_s$. This implies that μ^{t-1} is not stable at $t-1$, a contradiction. So, $|V_s| \leq |U_s|$ for each s. Hence $U_s \neq \emptyset$ implies $|U_s| = |V_s|$.

Let $\widetilde{I} = \cup_s U_s$, and define $\widetilde{\mu}$ as the matching such that

- for each $i \in \widetilde{I}$, $\widetilde{\mu}(i) = \mu^{t-1}(i)$,
- for each $i \notin \widetilde{I}$, $\widetilde{\mu}(i) = \mu^t(i)$.

Note that for each s, $\widetilde{\mu}(s)$ is obtained by replacing $|U_s|$ students by $|V_s|$ students. Since $|V_s| = |U_s|$, school s is matched to $|\widetilde{\mu}(s)| = |\mu^t(s)|$ different students under $\widetilde{\mu}$. As for the students, observe that, for each $i \in \cup_s U_s$, there exists only one school s' such that $i \in U_{s'}$ and only one school s'' such that $i \in V_{s''}$. So, each student $i \in I^t$ is matched to at most one school under $\widetilde{\mu}$, therefore $\widetilde{\mu}$ is a matching for (I^t, P^t, q^t).

We claim that $\widetilde{\mu}$ is stable at t. To see this, suppose that there exists a blocking pair, say (i, s). So, $s P_i^t \widetilde{\mu}(i) R_i^t \mu^t(i)$ and there exists $j \in \widetilde{\mu}(s)$ such that $i P_s j$. If $\widetilde{\mu}(s) = \mu^t(s)$, that is,

$\mu^t(s) \cap \widetilde{I} = \emptyset$, then (i,s) is also a blocking pair for μ^t at t, a contradiction. So, we can assume that $j \in \widetilde{I}$. Suppose first that $i \in \widetilde{I}$. Note that $\widetilde{\mu}(i) = \mu^{t-1}(i)$, so by B-refitting we must have $sP_i^{t-1}\mu^{t-1}(i)$. Since $j \in \widetilde{I}$, we have $\widetilde{\mu}(j) = \mu^{t-1}(j)$. So i has justified envy against j at $t-1$, i.e., (i,s) block μ^{t-1} at $t-1$, a contradiction. Hence $i \notin \widetilde{I}$. So $\mu^t(i)R_i^t\mu^{t-1}(i)$ and thus $sP_i^t\mu^{t-1}(i)$. Using assumption B this implies $sP_i^{t-1}\mu^{t-1}(i)$. Since $\widetilde{\mu}(j) = \mu^{t-1}(j)$ student i has thus justified envy agains j at $t-1$, which contradicts the stability of μ^{t-1} at $t-1$.

So at t we have two stable matchings, $\widetilde{\mu}$ and μ^t. By construction, for each student $i \in \widetilde{I}$, $\widetilde{\mu}P_i^t\mu^t$, and $\mu^t(i) = \widetilde{\mu}(i)$ if $i \notin \widetilde{I}$. So μ^t cannot be the Student-Optimal Matching at t, a contradiction. So, $U_s = \emptyset$ for each $s \in S$, which completes the proof. ■

Suppose that the outcome $\bar{\mu}$ of $\mathcal{M}^{SOSM}$ is not stable over early matches. Using the same arguments as in the beginning of the proof of Theorem 1 there exists i such that $sP_i^T\mu^T(i)$ and j such that iP_sj and $\bar{\mu}(j) = s$, with $t_j < T$ and $t_i = T$.

Since iP_sj and μ^{t_j} is stable at t_j we have $\mu^{t_j}(i)R_i^{t_j}s$. So, under assumption B, $\mu^{t_j}(i)R_i^{t_j+1}s$. By Lemma 3, $\mu^{t_j+1}(i)R_i^{t_j+1}\mu^{t_j}(j)$ and thus $\mu^{t_j+1}(i)R_i^{t_j+1}s$. Again by B-refitting, $\mu^{t_j+1}(i)R_i^{t_j+2}s$, and using Lemma 3 we obtain $\mu^{t_j+2}(i)R_i^{t_j+2}\mu^{t_j+1}(i)$. So, $\mu^{t_j+2}(i)R_i^{t_j+2}s$. Continuing this way, we eventually end up with $\mu^T(i)R_i^Ts$, a contradiction. So $\bar{\mu}$ is stable with early matches.

(2.) $\Rightarrow$ (1.) We show now that B-refitting is a necessary condition for the early stability of the mechanism. Let us assume on the contrary that B-refitting is not satisfied. It means that, for some rankings $\tilde{P}, \tilde{P}'$ and some schools s, s' with $s\tilde{R}i$ it holds that $s\tilde{P}s'$; $s'\tilde{P}'i$ and $s'\tilde{P}'s$ together with $\tilde{P}' \in F(\tilde{P}, s)$.

We construct a school choice problem and paths of preferences such that $\mathcal{M}^{SOSM}$ is not stable over early matches. Let $\Gamma = \left(I^1, S^1, (P_i^1)_{i\in J}, (P_s, q_s^1)_{s\in S}\right)$ be that school choice problem. Without loss of generality, we consider that $q_s^1 = 1$ for every $s \in S$. Let $i \in I^1$ be a student such that $P_i^1 := \tilde{P}$ and such that he declines any proposal, at the first stage and resubmits the new preferences that agree with F in the second stage: $P_i^2 := \tilde{P}'$.

We construct P_{-i}^1 and P_S such that $\mu^1(i) = s$ (s can be i) and $|\mu(s')| = 1$. It is always possible to construct such profiles if the number of students is greater than the number of schools in the school choice problem we construct. For instance, suppose that $\tilde{P} := [s_1, s_2, s_3, s, s', i]$ (the same reasoning applies for the other configurations). Let i_1, i_2, i_3 be respectively ranked first by schools s_1, s_2, s_3 and conversely. In addition we assume that $P_{s'} := [i, i', \ldots]$ and $P_{i'}^1 := [s', \ldots]$.

Clearly, the matching obtained in the first stage satisfies $\mu^1(i) = s$ and $\mu^1(i') = s'$.

Suppose that every student but student i accepts the proposal at stage 1 (including those who get nothing) then $I^2 = \{i\}$ and $\mu^2(i) = s$. If student i accepts the proposal at stage 2 then the final outcome ν is clearly not stable over early matches since i has envy against the pair (i', s') according to his latest submitted preferences $P_i^2 = \tilde{P}'$ which is such that $s' P_i^2 s$, together with $i P_s i'$. We obtain a contradiction, which proves the result.

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