A CONTINUUM MODEL FOR THE TOP TRADING CYCLES MECHANISM

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TOP TRADING CYCLES (TTC)

- Mechanism for allocating objects without transfers
- Used for school choice in New Orleans (until 2015) and San Francisco
- Pareto efficient, strategy-proof

TTC IN STANDARD SCHOOL CHOICE SETTING

- Colleges $c \in \{c_1, c_2, \dots, c_n\}$ with capacity q_c
- Students $s \in \{s_1, s_2, \dots, s_N\}$ are assigned to at most one college
- Colleges have strict (responsive) preferences over students and being unmatched, students have strict preferences over colleges

Colleges

c_1	c_2	c_3
s_1	s_2	s_3
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

Students

Student	$\succ$
s_1	$c_2, \dots$
s_2	$c_1, \dots$
s_3	$c_3, \dots$
s_4	$c_2, \dots$
s_5	$c_2, \dots$
s_6	$c_3, \dots$

TTC IN STANDARD SETTING

1. Colleges point to their favorite student, students point to their favorite college

Colleges

c_1	c_2	c_3
s_1	s_2	s_3
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

Students

Student	$\succ$
s_1	$c_2, \ldots$
s_2	$c_1, \ldots$
s_3	$c_3, \ldots$
s_4	$c_2, \ldots$
s_5	$c_2, \ldots$
s_6	$c_3, \ldots$

TTC IN STANDARD SETTING

2. Algorithm selects a union of cycles

Colleges

c_1	c_2	c_3
s_1	s_2	s_3
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

Students

Student	$\succ$
s_1	$c_2, \ldots$
s_2	$c_1, \ldots$
s_3	$c_3, \ldots$
s_4	$c_2, \ldots$
s_5	$c_2, \ldots$
s_6	$c_3, \ldots$

TTC IN STANDARD SETTING

3. 'Clear' cycles by assigning students to the college they are pointing to

Colleges

c_1	c_2	c_3
$s_1 \rightarrow c_2$	$s_2 \rightarrow c_1$	$s_3 \rightarrow c_3$
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

Students

Student	$\succ$
s_1	$c_2, \ldots$
s_2	$c_1, \ldots$
s_3	$c_3, \ldots$
s_4	$c_2, \ldots$
s_5	$c_2, \ldots$
s_6	$c_3, \ldots$

TTC IN STANDARD SETTING

4. Remove assigned students. Update college capacities. Recurse.

Colleges

c_1	c_2	c_3
$s_1 \rightarrow c_2$	$s_2 \rightarrow c_1$	$s_3 \rightarrow c_3$
~~s_2~~	s_4	s_5
~~s_3~~	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	~~s_1~~	~~s_1~~

Students

Student	$\succ$
s_1	$c_2, \ldots$
s_2	$c_1, \ldots$
s_3	$c_3, \ldots$
s_4	$c_2, \ldots$
s_5	$c_2, \ldots$
s_6	$c_3, \ldots$

TTC IN STANDARD SETTING

Question: Where will TTC assign a given student?

Do we have a simple way to provide an answer?

PRIORITIES CAN BE TRADED

Colleges

c_1	c_2	c_3
$s_1 \rightarrow c_2$	$s_2 \rightarrow c_1$	$s_3 \rightarrow c_3$
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

- TTC is recursive
- Priority at one college can lead to early assignment in other colleges
- E.g. s_1 gets into c_2 even though lowest ranked

OUTLINE OF TALK

1. Develop a tractable model for TTC
2. Characterize structure of TTC allocations
3. Perform comparative statics

RELATED LITERATURE

- School choice as a mechanism design problem
  - Abdulkadiroglu & Sönmez (2003)
- Characterizations of TTC
  - Ma (1994), Morrill (2013)
  - Papai (2000), Pycia & Ünver (2016)
  - Abdulkadiroglu & Che (2010), Dur (2012)
- Large markets in school choice
  - Che & Tercieux (2015), Azevedo & Leshno (2016)

A CONTINUUM FRAMEWORK FOR TTC

WHY LARGE MARKETS?

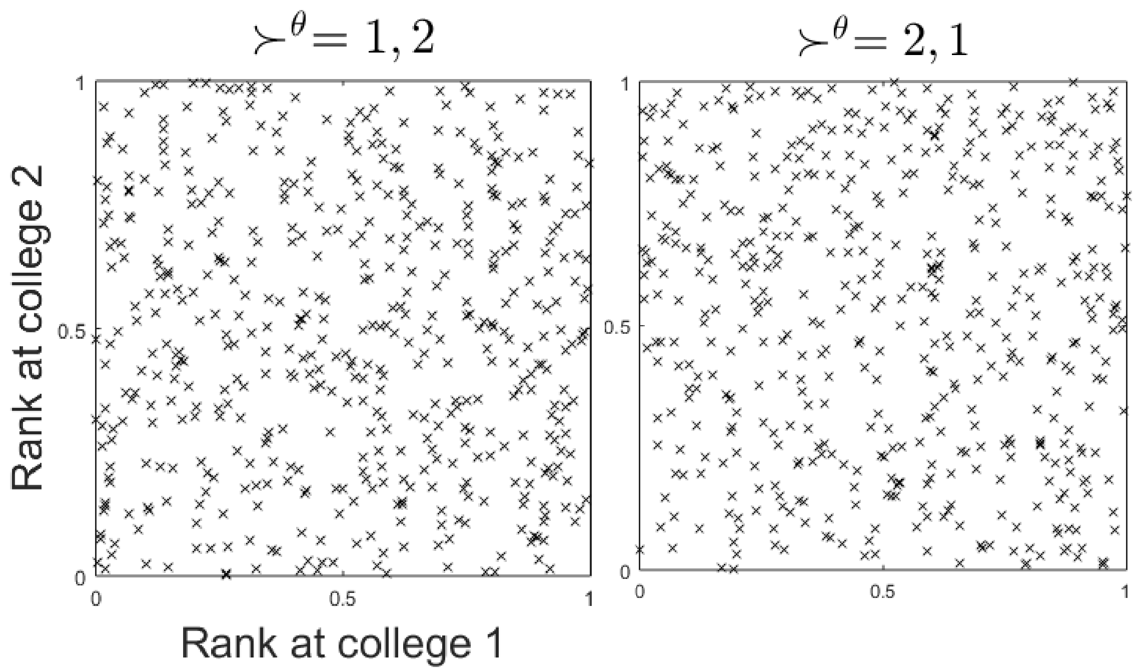
- Simplified analysis of Deferred Acceptance (DA)
  - Azevedo & Leshno (2016) analyze deferred acceptance in the large market setting, where the number of colleges is fixed and the number of students goes to infinity
  - Show that the outcome of DA in large markets has a simple description
- Regime where TTC and DA are similarly well-behaved
  - Che & Tercieux (2015) showed that TTC and DA are approximately efficient and stable in large markets when agent preferences are uncorrelated
- We give a simple description of the outcome of TTC in large markets.

THE CONTINUUM MODEL (AZEVEDO & LESHNO 2016)

- Colleges $c \in \{c_1, \dots, c_n\} = C$
  - Capacities q_c
- Student types $\theta = \left(\succ^\theta, r^\theta\right) \in \Theta$
  - $\succ^\theta$ is the student's strict preferences over colleges
  - $r_c^\theta \in [0,1]$ is the student's rank at college c, where *lower* ranks are preferred

THE CONTINUUM MODEL (AZEVEDO & LESHNO 2016)

- An **economy** $E = (C, \Theta, \eta, q)$
  - C is the set of colleges
  - q is college capacities
  - Θ is the set of student types
  - η is a probability measure over student types



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ALLOCATIONS

- An allocation is a function

$$\mu : \Theta \rightarrow C \cup \emptyset$$

satisfying

1. (Measurability) $\mu^{-1}(c)$ is η-measurable for each college $c \in C$
2. (Respects capacities) $\eta\left(\mu^{-1}(c)\right) \leq q_c$ for each college $c \in C$
3. (Right continuous) For any sequence of student types $\theta^k = (\succ, r^k)$ and $\theta = (\succ, r)$ with r^k converging to r and $r_c^k \geq r_c^{k+1} \geq r_c$, there is some large K such that $\mu\left(\theta^k\right) = \mu(\theta)$ for $k > K$.

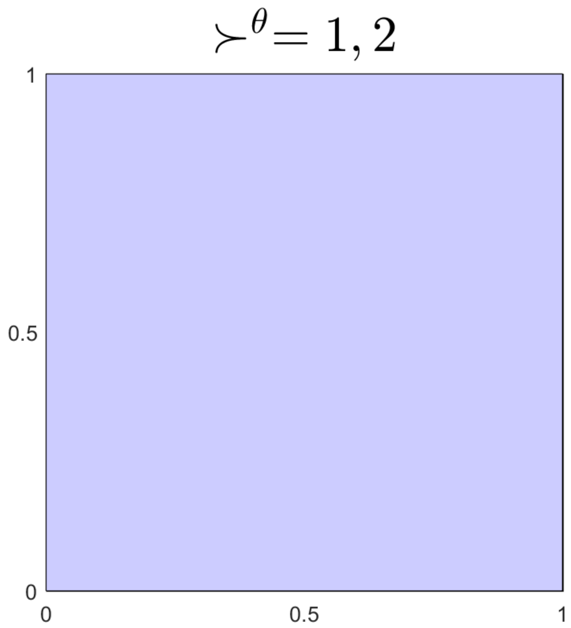
CHALLENGES TO DEFINING TTC IN THE CONTINUUM MODEL

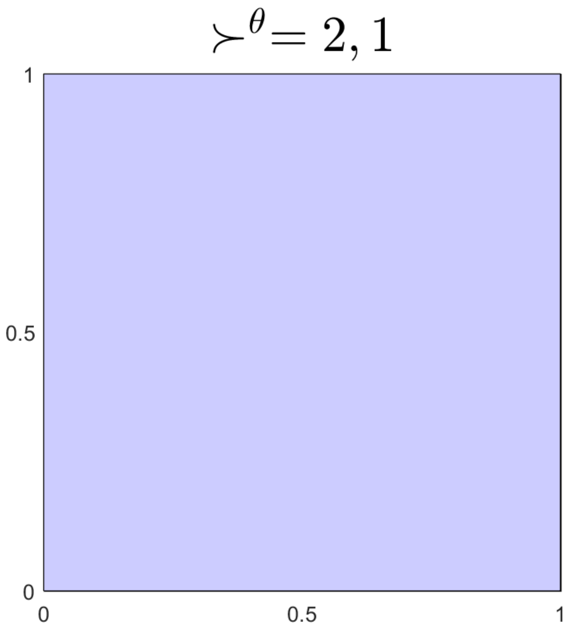
- Multiplicity in pointing
  - Colleges give priority to a continuum of students
  - Definition of continuum cycles?
  - In the discrete setting, when colleges give priority to multiple students, outcome can depend on cycle selection rule (e.g. Kesten 2006)
- Tracking the progression of the algorithm

ASSUMPTIONS

- Assumptions on economy $E = (C, \Theta, \eta, q)$:
  1) (Lipschitz density) Distribution η has a density ν that is Lipschitz continuous except on a finite grid, and bounded from above, $\eta(A) = \int_A \nu(\theta) d\theta \;\; \forall\, A \subset \Theta$
  2) (Strict preferences) Colleges' indifference curves have measure 0,
  $$\eta\left(\{\theta \in \Theta : r_c^\theta = x\}\right) = 0 \;\; \forall x, c$$
  3) (Percentile ranks) Student ranks are equal to their percentile ranks
  $$\eta\left(\theta : r_c^\theta \leq x\right) = x \;\; \forall x$$

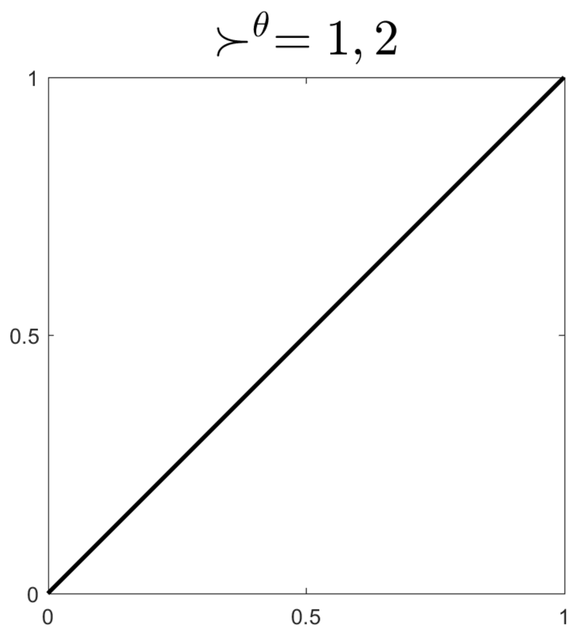
EXAMPLE – UNIFORM

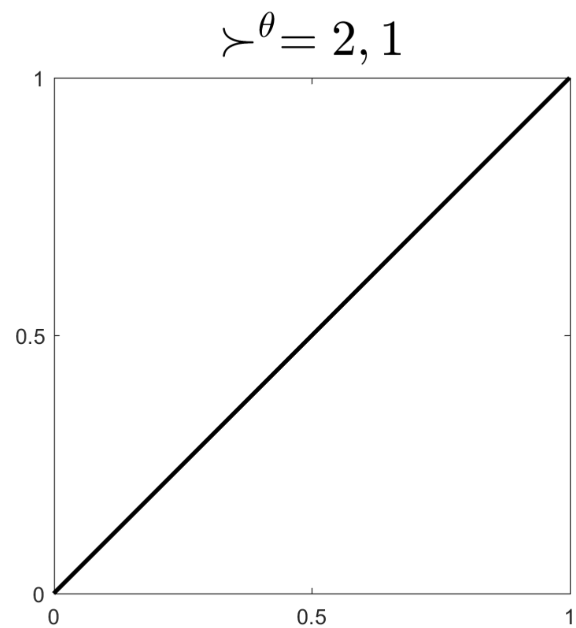




Satisfies assumptions (1) – (3)

EXAMPLE – PERFECTLY CORRELATED





Does not satisfy assumption (1) (Lipschitz density)

RESULTS

MAIN RESULT

- We show that there exist n^2 cutoffs $p_j^{(i)}$ such that the TTC allocation μ is given by

$$\mu(\theta) = \max_{\succ^\theta}\{c: \exists j \; s.t. \, r_j^\theta \geq p_j^{(c)}\}.$$

Interpretation of cutoffs $p_j^{(i)}$:

'How much does college j have to like me for me to be able to trade a seat at college j for a seat at college i?'

- We provide a method for calculating the cutoffs $\mathrm{p}_\mathrm{j}^{(\mathrm{i})}$ as solutions to a differential equation

PROOF OF MAIN RESULT – A ROADMAP

1. Define TTC in the continuum
   - Limit of discrete trading cycles

2. Represent TTC in the continuum by a 'TTC Path'

3. Characterize the TTC allocation
   - Interpret the TTC allocation using cutoffs
   - Use connection to Markov Chains to show uniqueness

DEFINING TTC IN THE CONTINUUM MODEL

1. Intuition:
   - Recursively allow discrete cycles of top priority students to trade their seats
   - Take the limit as the measure of students in each cycle goes to zero

2. Properties of Top Trading Cycles:
   - TTC assigns top priority students first
   - TTC satisfies 'trade balance' – the number of students offered seats by a college is the same as the number of students assigned a seat at the college

TTC ASSIGNS TOP PRIORITY STUDENTS FIRST

γ_i : Top rank of unassigned student at college c_i

Colleges

c_1	c_2	c_3
s_1	s_2	s_3
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

TTC ASSIGNS TOP PRIORITY STUDENTS FIRST

γ_i : Top rank of unassigned student at college c_i

Colleges

c_1	c_2	c_3
$\gamma_1 = 1 \rightarrow s_1$	$\gamma_2 = 1 \rightarrow s_2$	$\gamma_3 = 1 \rightarrow s_3$
s_2	s_4	s_5
s_3	s_5	s_6
$\vdots$	$\vdots$	$\vdots$
s_4	s_1	s_1

- Time $t = 0$

TTC ASSIGNS TOP PRIORITY STUDENTS FIRST

γ_i : Top rank of unassigned student at college c_i

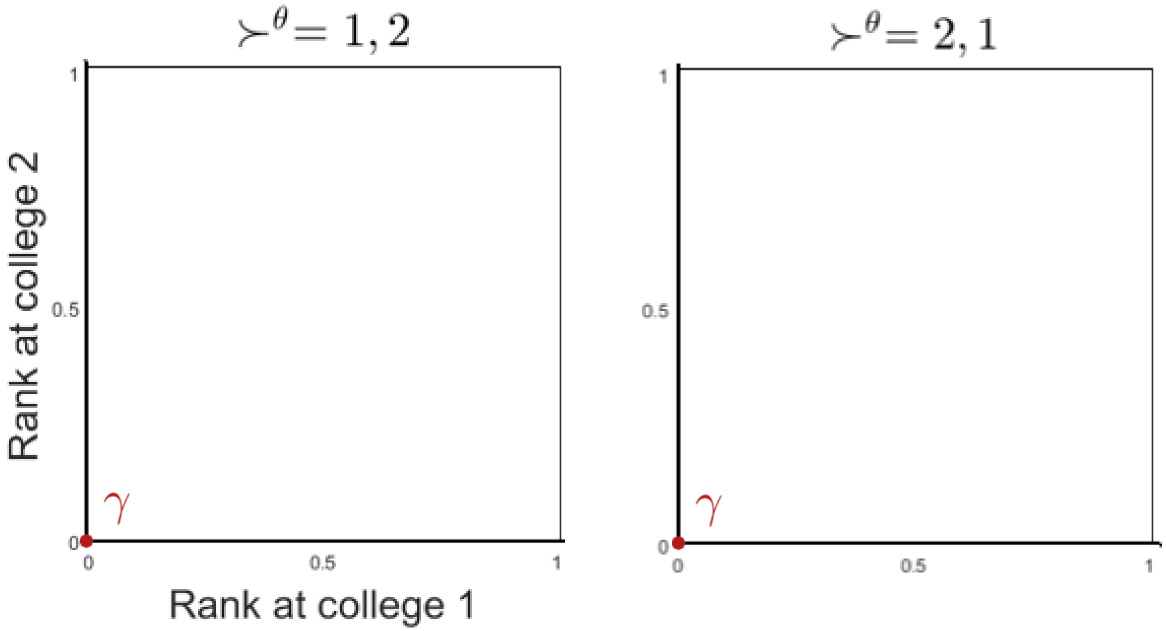
Colleges

c_1	c_2	c_3
~~s_1~~	~~s_2~~	~~s_3~~
~~s_2~~	$\gamma_2 = 2 \rightarrow s_4$	$\gamma_3 = 2 \rightarrow s_5$
~~s_3~~	s_5	s_6
$\gamma_1 = 4 \rightarrow \vdots$	$\vdots$	$\vdots$
s_4	~~s_1~~	~~s_1~~

- Time $t = 1$

TTC ASSIGNS TOP PRIORITY STUDENTS FIRST

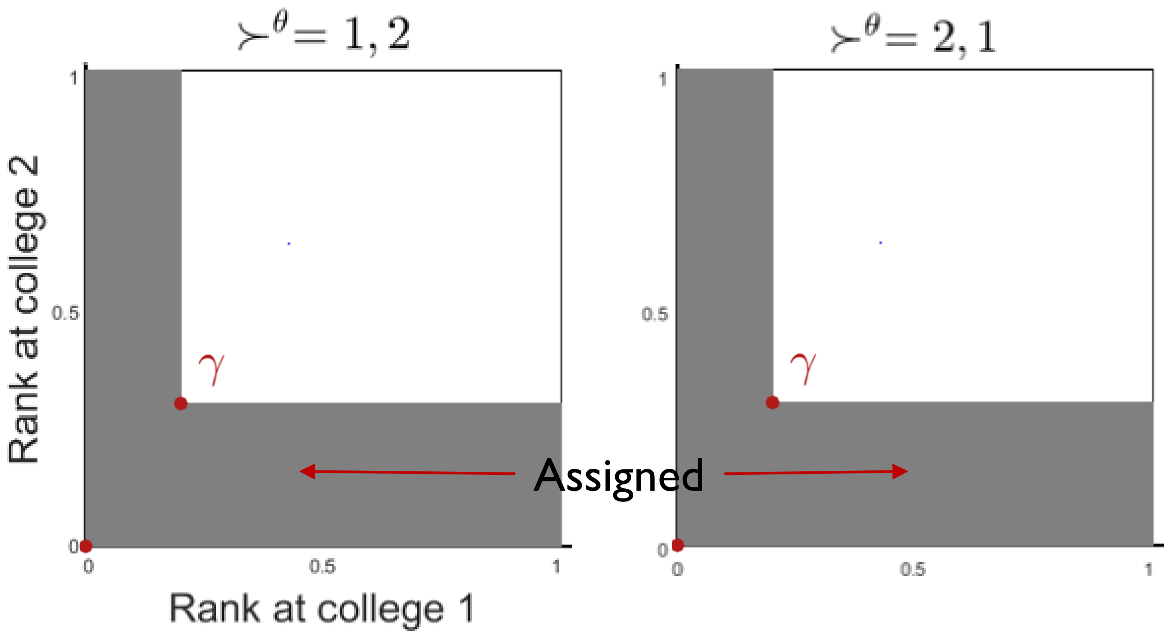
γ_i : Top rank of unassigned student at college c_i



- Time $t = 0$

TTC ASSIGNS TOP PRIORITY STUDENTS FIRST

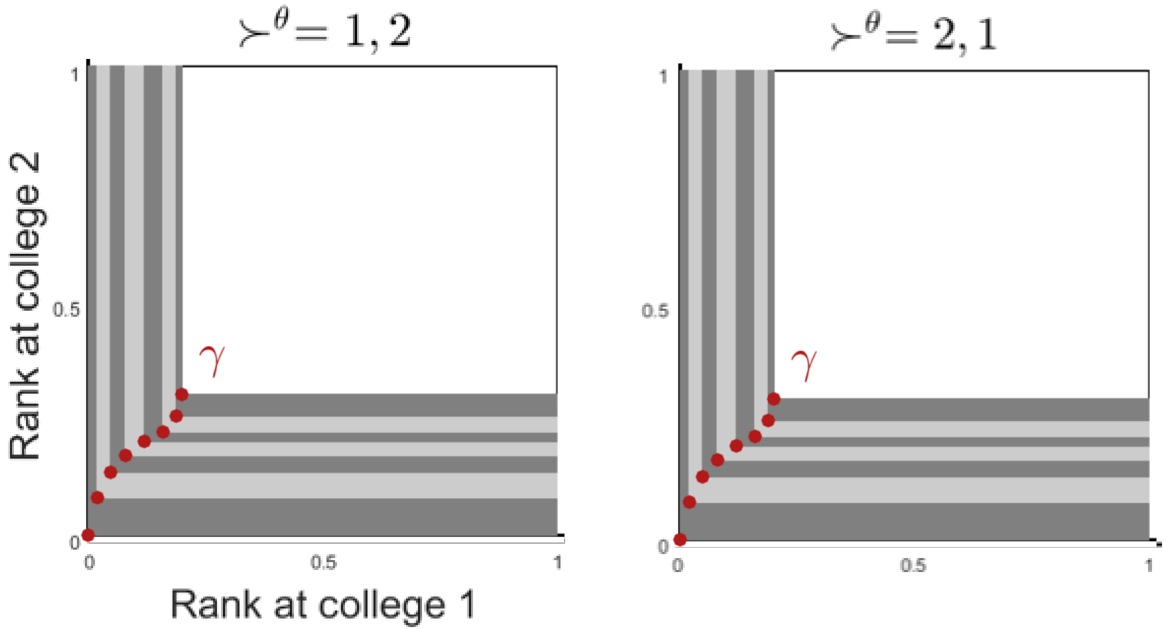
γ_i : Top rank of unassigned student at college c_i



- Time $t = t_1$

LIMIT OF DISCRETE TRADING CYCLES

γ_i : Top rank of unassigned student at college c_i



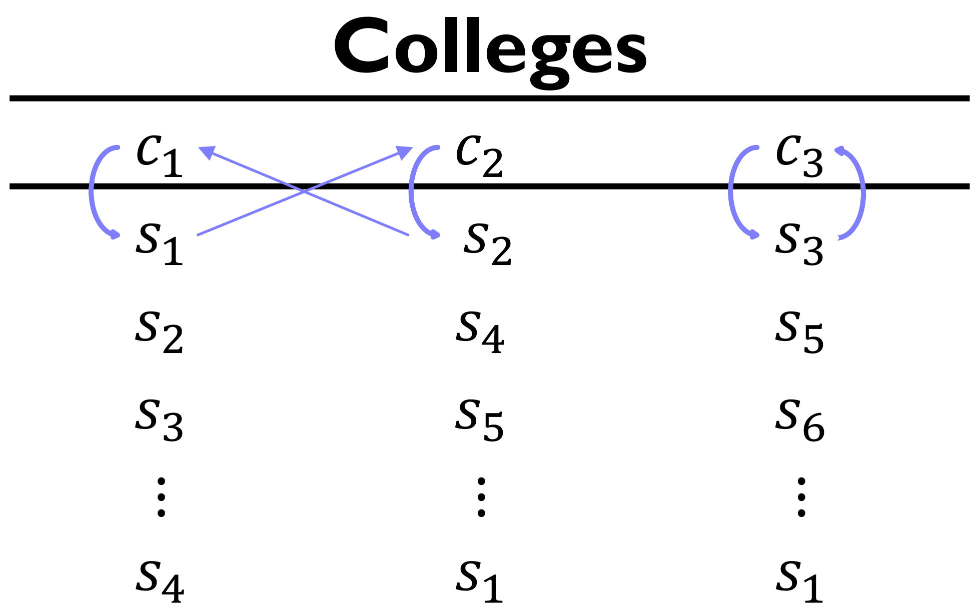
- Time $t = 0$ to $t = t_1$

TTC SATISFIES TRADE BALANCE EQUATIONS

- The following trade balance equations hold:

$$\textit{Measure of students offered a seat at } c = \textit{Measure of students assigned a seat at } c$$

- E.g. c_1 offers 1 seat (to s_1) and assigns 1 seat (to s_2)

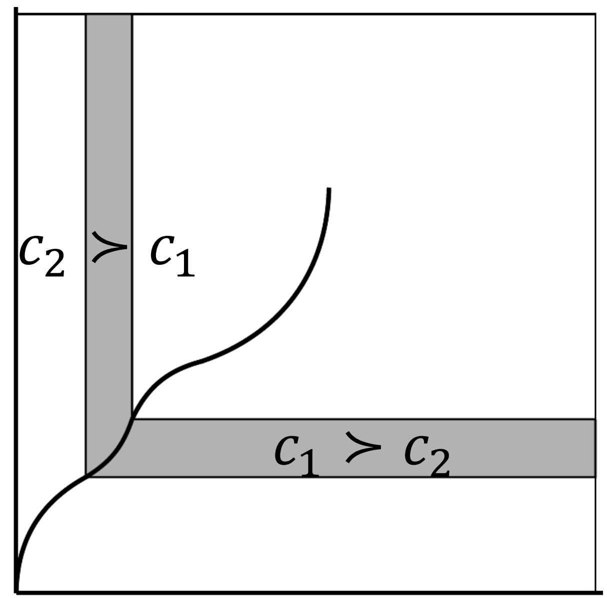


CONTINUUM TTC = TTC PATH

- A **TTC path** is a function γ that is
  - Continuous
  - Piecewise differentiable
  - Increasing and strictly increasing in at least one coordinate

and satisfies the **trade balance equations**

$$\begin{matrix} Measure\ of\ students \\ offered\ a\ seat\ at\ c \end{matrix} = \begin{matrix} Measure\ of\ students \\ assigned\ a\ seat\ at\ c \end{matrix}$$



Lemma:

Every instance of TTC in the continuum is represented by a TTC path

EXISTENCE OF TTC PATH

Theorem (Existence)

Suppose that η satisfies the following assumptions:

- (Lipschitz density) η has a density ν that is Lipschitz continuous except on a finite grid, and bounded from above,

$$\eta(A) = \int_A \nu(\theta) d\theta \quad \forall\, A \subset \Theta$$

- (Strict preferences) Colleges' indifference curves have measure 0,

$$\eta\left(\{\theta \in \Theta : r_c^\theta = x\}\right) = 0 \quad \forall x, c$$

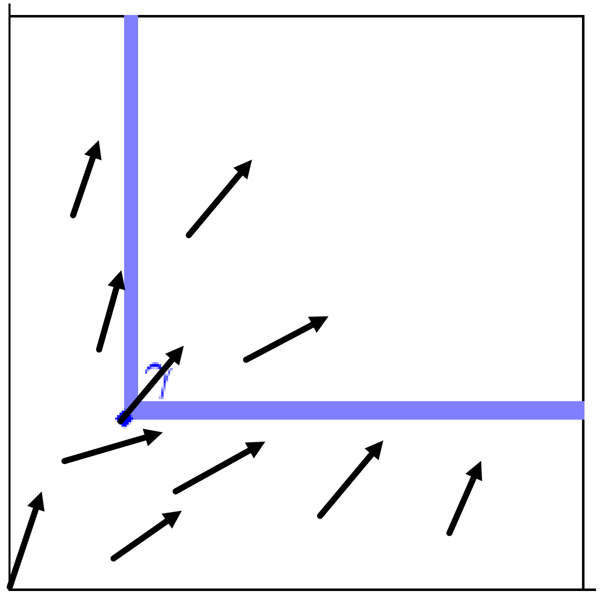
- (Percentile ranks) Student ranks are equal to their percentile ranks

$$\eta\left(\theta : r_c^\theta \leq x\right) = x \quad \forall x$$

Then there exists a TTC path.

PROOF – EXISTENCE OF TTC PATH

- The trade balance equations give a gradient for the TTC path at each point
- A path that starts at the origin and follows the gradient vector field will be a TTC path
- Solution to the 'Initial value problem'
- Existence given by Picard-Lindelöf when gradients are Lipschitz continuous



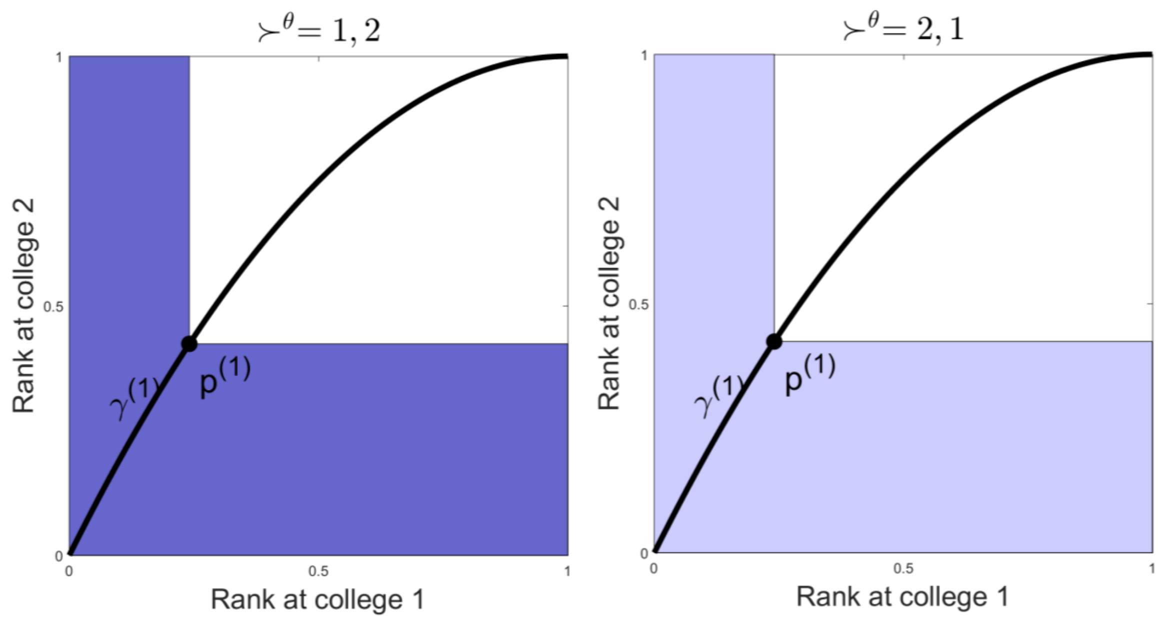
TTC ALLOCATION

- Students are assigned in 'rounds', tracked by TTC path γ
- In each round, assigned students receive their favorite college that is available in that round
- A round ends when a college reaches capacity, giving cutoffs p

TTC path γ → TTC cutoff p → TTC allocation $TTC(\gamma)$

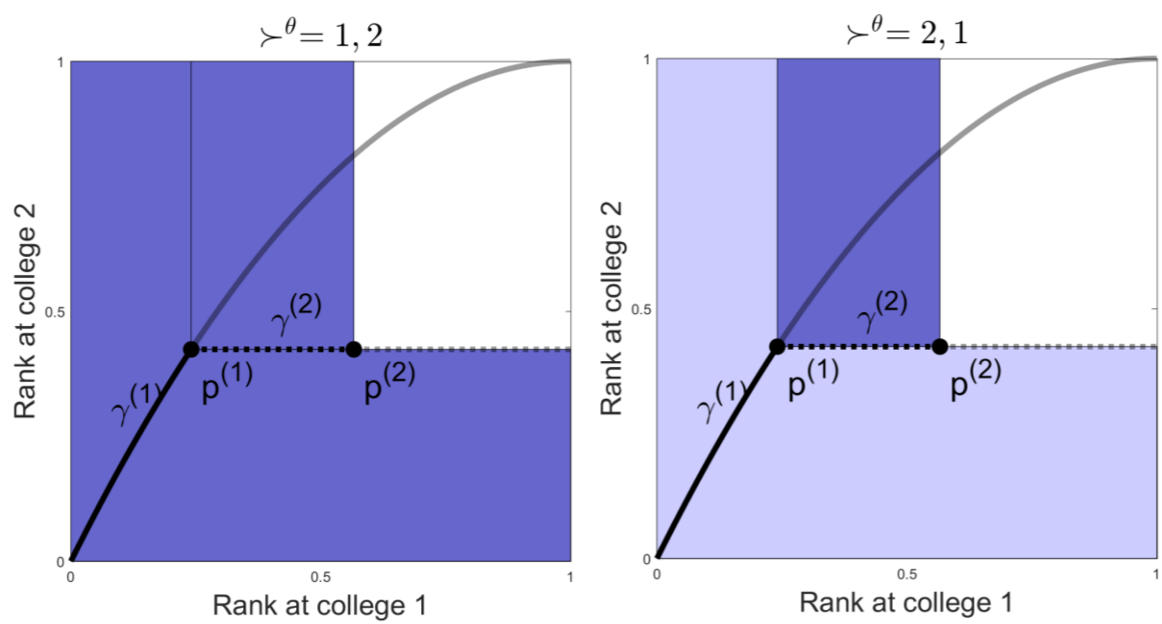
TTC ALLOCATION

- E.g. $n = 2$
  - Students: Uniform ranks, twice as many want college 2 as want college 1
  - Round 1, $\gamma^{(1)}$, $p^{(1)}$
  - Available colleges: {1,2}



TTC ALLOCATION

- E.g. $n = 2$
  - Students: Uniform ranks, twice as many want college 2 as want college 1
  - Round 2, $\gamma^{(2)}, p^{(2)}$
  - Available colleges: {1}



TTC ALLOCATION – CUTOFF CHARACTERIZATION

Theorem (Cutoffs)

There exist n^2 cutoffs $p_j^{(i)}$ such that the TTC allocation μ is given by

$$\mu(\theta) = \max_{\succ^\theta}\{c: \exists j \ s.t. r_j^\theta \geq p_j^{(c)}\}.$$

Moreover, the cutoffs $p^{(i)}$ are the endpoints of n recursive TTC paths,

$$p^{(i)} = \gamma^{(i)}\left(t^{(i)}\right)$$

Interpretation of cutoffs $p_j^{(i)}$: *'How much does college j have to like me for me to be able to trade a seat at college j for a seat at college i?'*

UNIQUENESS OF TTC ALLOCATION

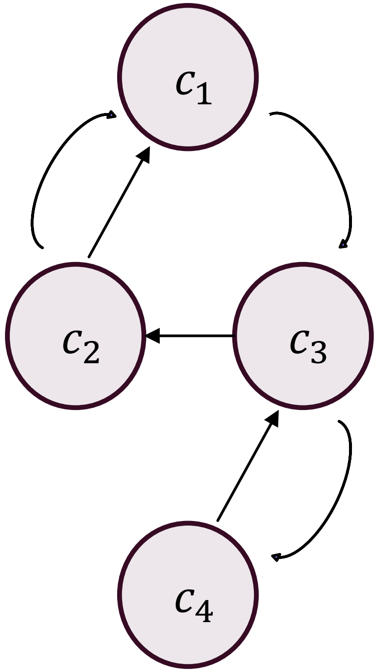
Theorem

Let $E = (C, \Theta, \eta, q)$ be a continuum economy. Suppose that η satisfies assumptions (1)-(3), and suppose that γ and γ' are TTC paths for the economy E.

Then the allocations obtained by $TTC(\gamma)$ and $TTC(\gamma')$ differ on a set of students of η-measure 0.

PROOF TECHNIQUE – CONNECTION TO MARKOV CHAINS

- Markov chain:
  - State i : College c_i
  - Transition probability p_{ij} :
    Probability that a student offered a seat by college c_i wants college c_j
- Trading cycles correspond to positive recurrent communication classes
  - Cycle → communication class
  - Trade balance equations → positive recurrent



PROOF SKETCH

- Sources of multiplicity in TTC path:
  - At what relative rates should disjoint trading cycles be cleared?
  - In what order should disjoint trading cycles be cleared?
  - Will these choices affect future cycles?

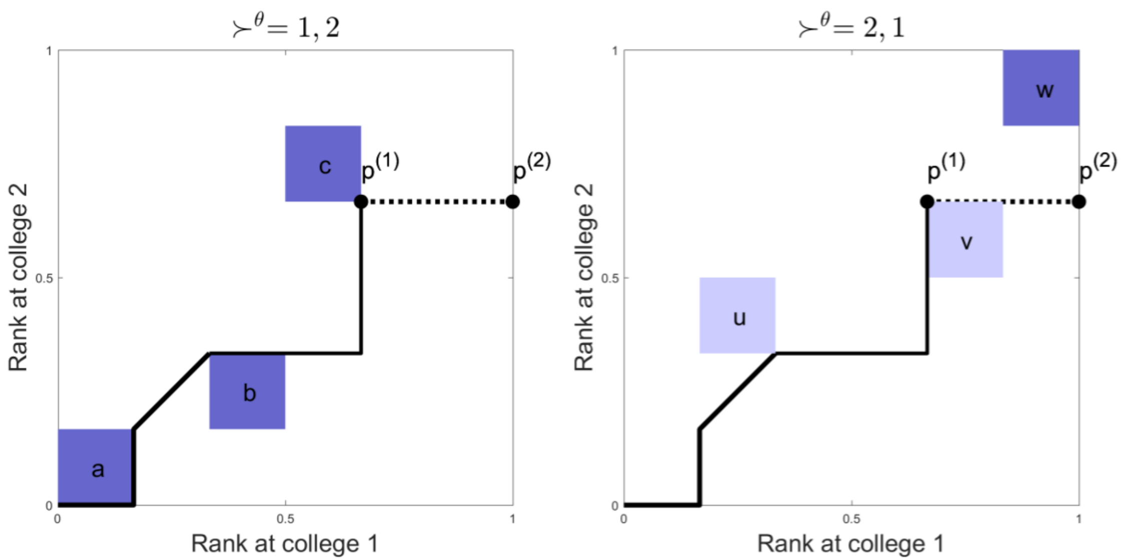
Key idea: Cycles persist

- Reduces to same intuition as discrete case:
  - The order in which the cycles are cleared does not matter
  - The same cycles are eventually cleared

CONVERGENCE

EMBEDDING DISCRETE TTC IN THE CONTINUUM MODEL

- $N = |S|$ students
- Each student s represented by cube in $\mathbb{R}^N$
- Student s assigned a seat in college c if and only if their cube is fully assigned to c.



EMBEDDING DISCRETE TTC IN THE CONTINUUM MODEL

Proposition

The outcome of TTC in the continuum embedding gives the same allocation as TTC in the discrete model.

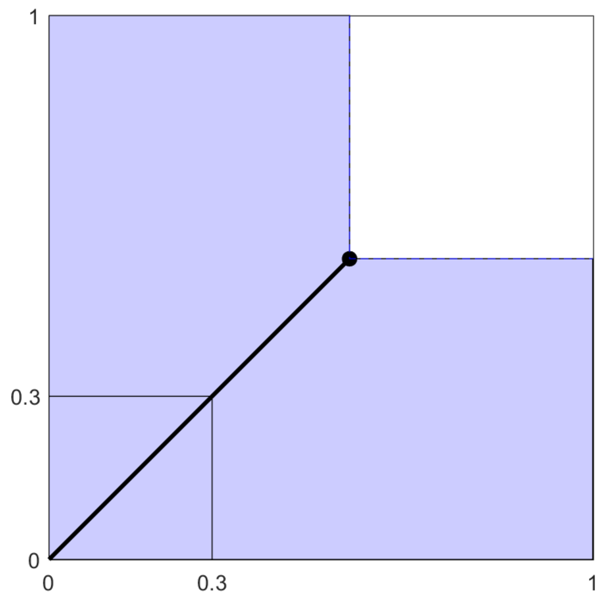
CONVERGENCE

Theorem

Let $E = (C, \Theta, \eta, q)$ and $\tilde{E} = (C, \theta, \tilde{\eta}, q)$ be two continuum economies, where η and $\tilde{\eta}$ satisfy assumptions (1)-(3), have full support, and have total variation distance ε. Then the TTC allocation on these two economies differ by on a set of students of measure $O(\varepsilon|C|^2)$.

APPLICATIONS

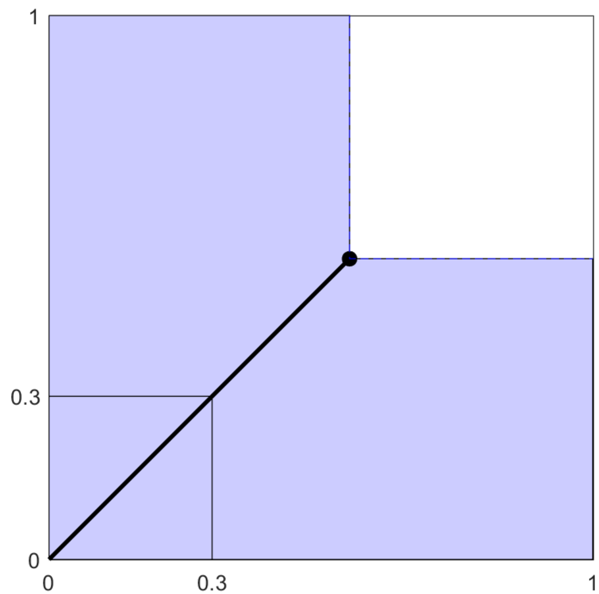
TTC PRIORITIES ARE 'BOSSY'



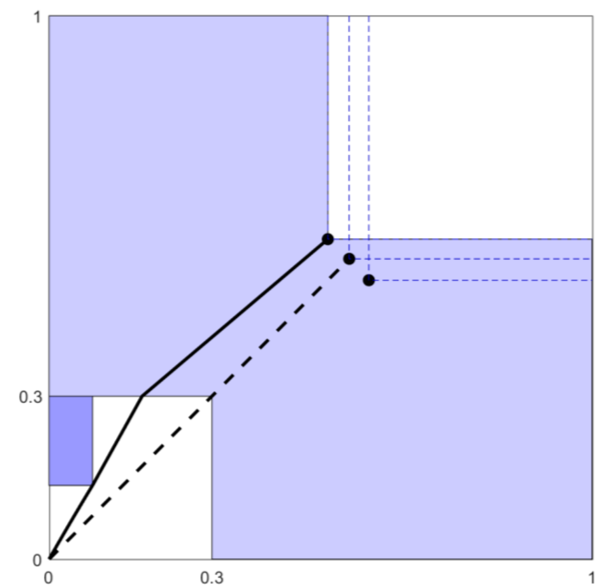
$q = 0.4, r = 0.3$

- Two colleges, uniform preferences, same capacity q
- Half the students prefer college 1, half the students prefer college 2
- Mechanism designer shifts priorities of 'top priority students'- those in the top r percentile at both colleges

TTC PRIORITIES ARE 'BOSSY'



$q = 0.4, r = 0.3$



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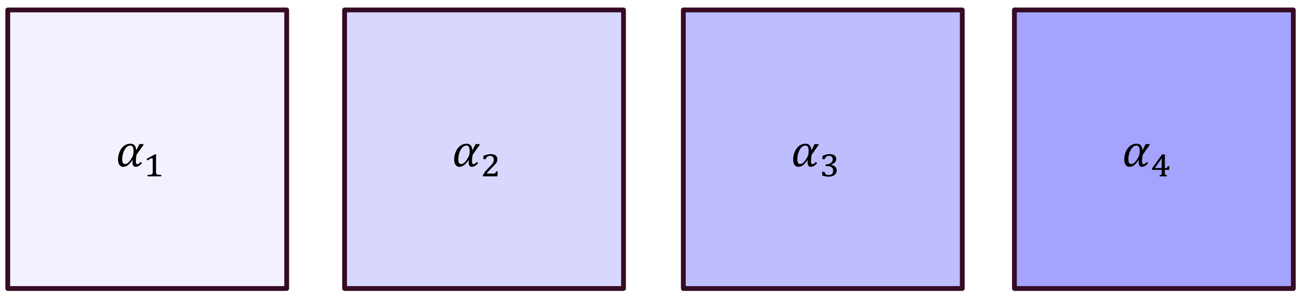
TTC PRIORITIES ARE 'BOSSY'

- Implications:
  - Tiebreaking of high priority students can affect allocations of other students without affecting the allocation of the high priority students
  - E.g. 'Equitable TTC' (Hakimov & Kesten 2014) allows college c to point to top q_c students, cycle clearing governed by a pointing rule. Choice of pointing rule is important.
  - Cutoffs cannot be computed directly – require recursive structure

COMPARATIVE STATICS- COLLEGE POPULARITY

- n colleges $C = \{c_1, c_2, \dots, c_n\}$
- For each college c_i, measure α_i of students with uniform ranks and top choice c_i

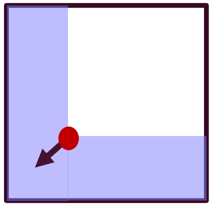


- Assume that c_1 is the most demanded college, $\frac{\alpha_1}{q_1} = \max_i \frac{\alpha_i}{q_i}$

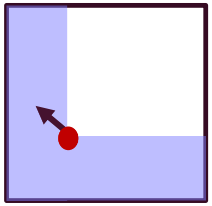
Question: Which students receive a seat at college c_1?

COMPARATIVE STATICS- COLLEGE POPULARITY

- Cutoffs for college 1 : $p_i^{(1)} = 1 - \left(1 - \frac{q_1}{\alpha_1}\right)^{\alpha_i}$
- Suppose college 1 increases its popularity α_1. How do the cutoffs change?
  - More students want college 1: $\boldsymbol{p_1^{(1)}} \downarrow \boldsymbol{p_i^{(1)}} \downarrow$

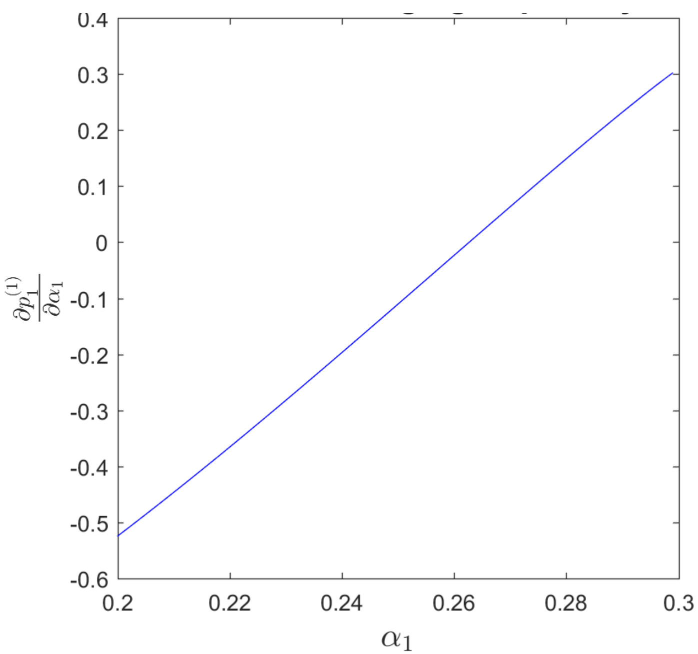


  - College 1 increases in popularity compared to other colleges: $\boldsymbol{p_1^{(1)}} \uparrow \boldsymbol{p_i^{(1)}} \downarrow$

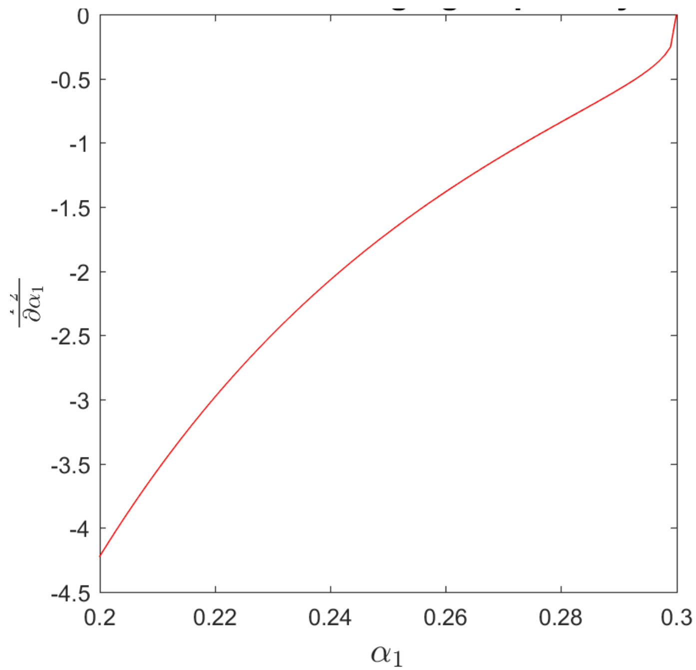


COLLEGE POPULARITY

$$\frac{\partial p_1^{(1)}}{\partial \alpha_1} = -\left(1-\frac{q_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{q_1}{\alpha_1-q_1}+\log(1-\frac{q_1}{\alpha_1})\right)^{\alpha_1}$$



$$\frac{\partial p_i^{(1)}}{\partial \alpha_1} = -\frac{\alpha_i q_1}{\alpha_1^2}\left(1-\frac{q_1}{\alpha_1}\right)^{-(1-\alpha_i)}$$



CONCLUSIONS

- Developed a tractable model for TTC
  - Assignment characterized by n^2 cutoffs, given by solution to differential equation
- Characterized structure of TTC allocations
  - For each pair of schools, there is cutoff for the lowest score at school i that can be traded for spot at school j
  - Used connection to Markov chains to show uniqueness
- Quantify the effects of changes on the allocation of TTC
  - TTC priorities are bossy
  - Comparative statics- College popularity

Thank You!

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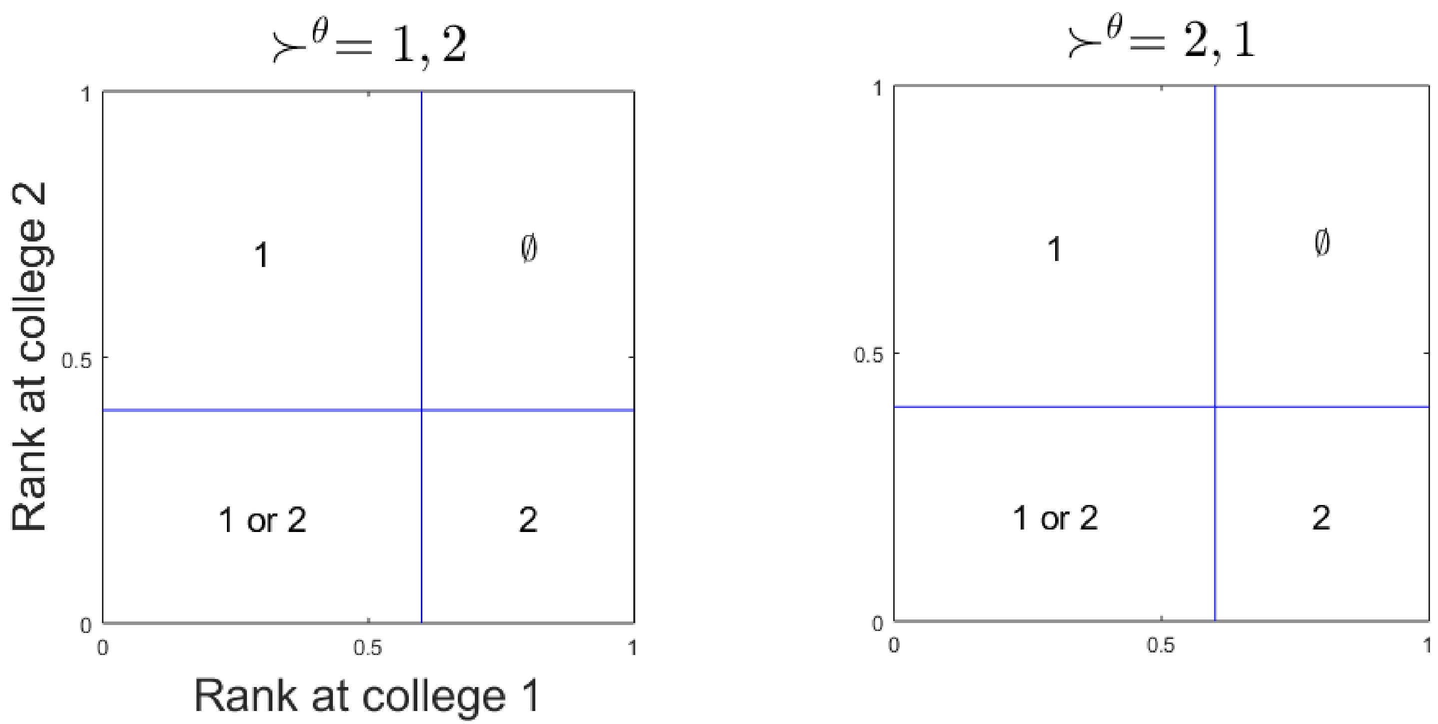
COMPARISON WITH DEFERRED ACCEPTANCE

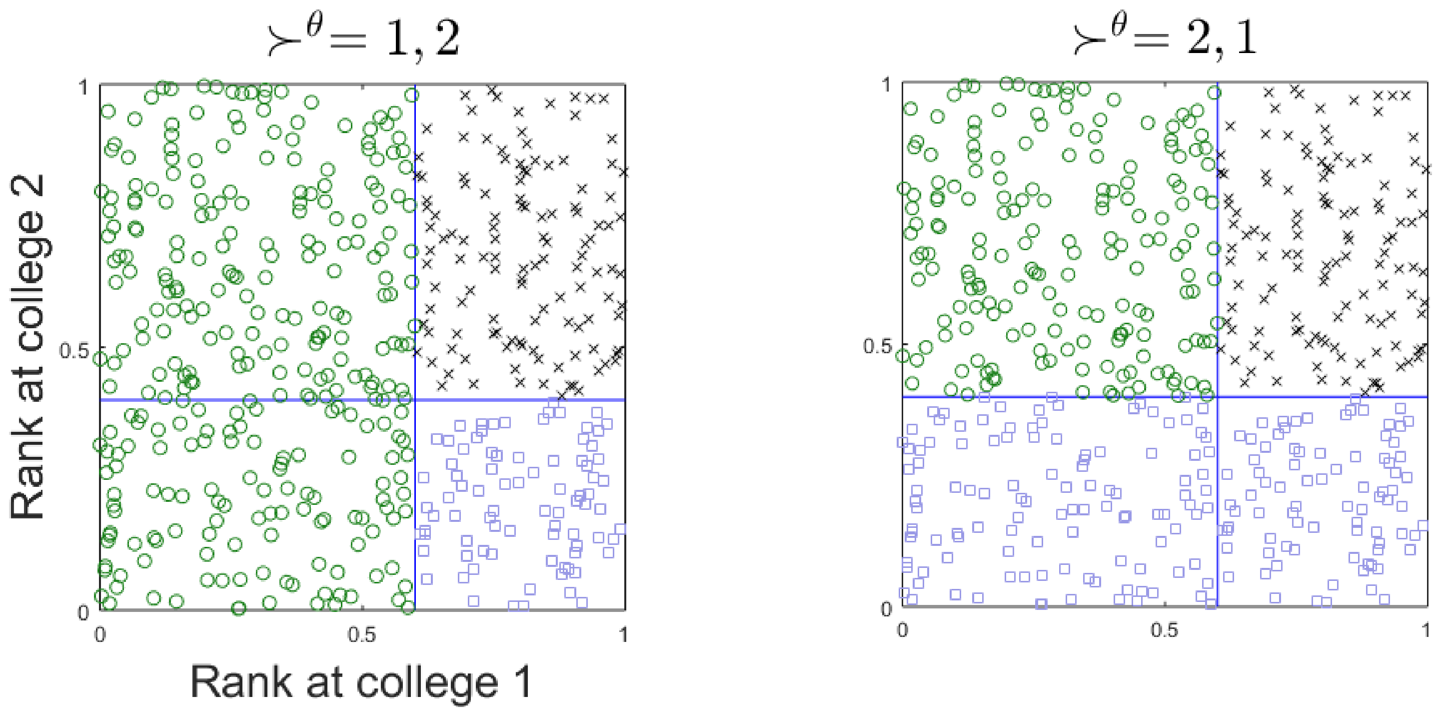
- Efficient vs Stable
- n^2 cutoffs vs n cutoffs
- Computed recursively vs Solution to n supply/demand equations

CHOICE SETS

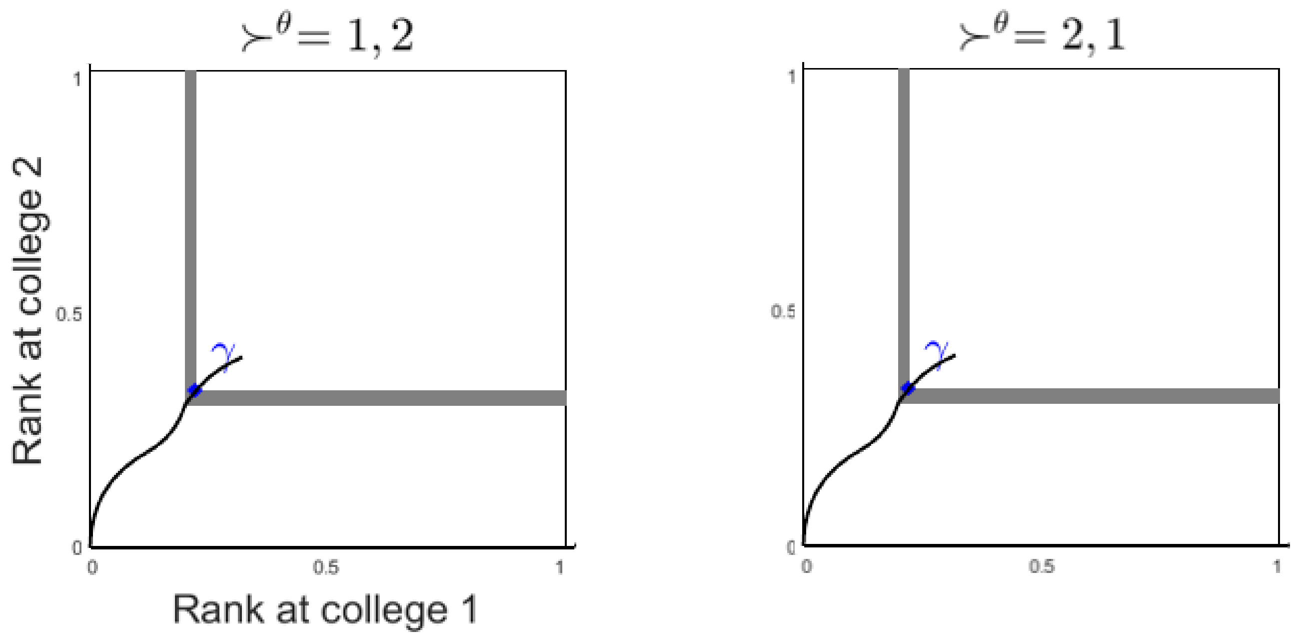


$\succ^{\theta} = 1, 2$
$\succ^{\theta} = 2, 1$
Rank at college 2
Rank at college 1
1
∅
1 or 2
2
1
∅
1 or 2
2
0
0.5
1

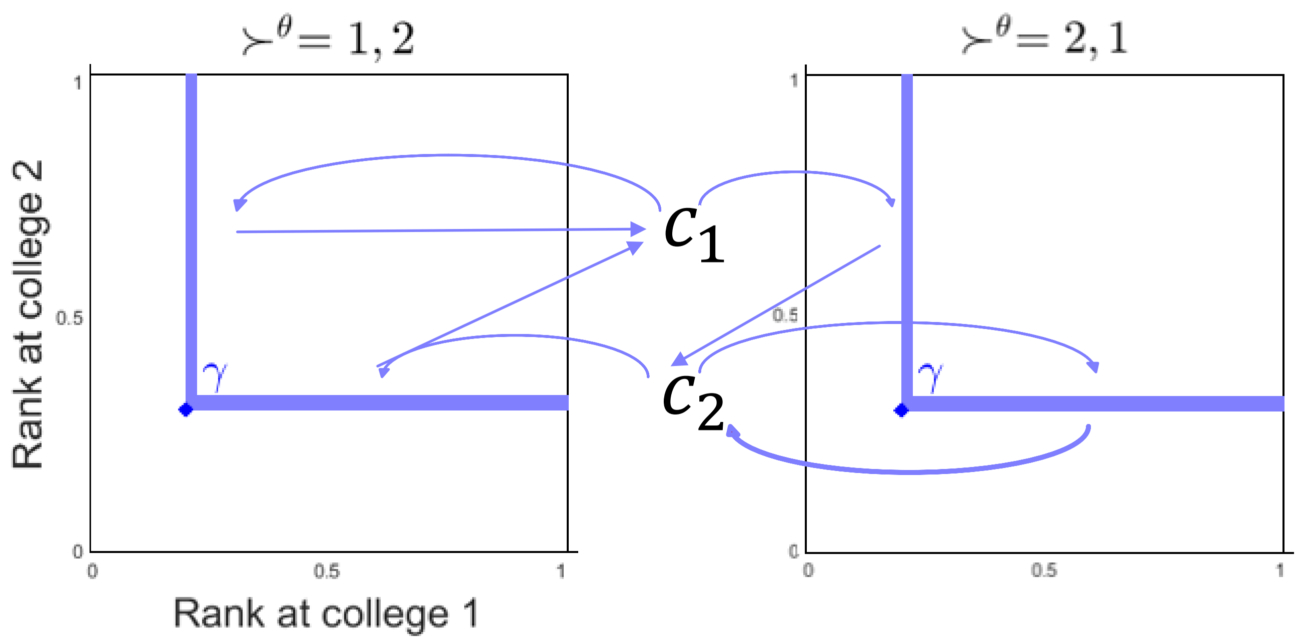
CHOICE SETS


$\succ^\theta = 1, 2$
$\succ^\theta = 2, 1$
Rank at college 2
Rank at college 1
0
0.5
1
0
0.5
1
0
0.5
1
0
0.5
1

CONTINUUM TTC



DEFINING TTC IN THE CONTINUUM MODEL



PROOF OF MAIN RESULT – A ROADMAP

1. Define TTC in the continuum
   - Limit of discrete trading cycles

2. Represent TTC in the continuum by a 'TTC Path'

3. Characterize the TTC allocation
   - Interpret the TTC allocation using cutoffs
   - Use connection to Markov Chains to show uniqueness

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